$BDMSO_2$ vs. MSO_2

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V WLOGIA
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Fragments of SO

- We investigate fragments of second-order quantification:

$$\exists R \phi(R)$$

"There is a relation R such that $\phi(R)$."

Expressing Problems with Sentences.

- 3-Colourability: Given a graph $G = (V, E)$ decide whether it can coloured with three different colours, so that no tow adjacent vertices has the same colour.
- It can be expressed in SO:

$$\exists R \exists G \exists B (\forall x (Rx \vee Gx \vee Bx) \wedge$$

$$\forall x \forall y ((Rx \wedge Ry) \vee (Gx \wedge Gy) \vee (Bx \wedge By) \rightarrow \neg Exy))$$

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Relations of Bounded Degree

- Let A be a set and $R \subset A^k$ a k-ary relation.
- The Gaifman graph of R is the graph $G = (A, E)$ such that

$$E = \{(a, a') \in A | \text{ there is } (a_1, \ldots, a_k) \in R,$$

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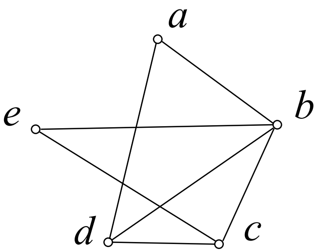
Quantifying relations of bounded degree.

$$R = \{(a,a,b),\ (b,c,e),\ (a,d,a),\ (d,b,c)\}$$



Bounded-Degree Second-Order Logic

- The Gaifman degree $d_{\mathcal{G}}(R)$ of R is the maximum degree of a vertex in G.
- The Bounded-Degree Second-Order Logic BDSO allows quantification over bounded-degree relations.
- The quantifiers

$$\exists^d \quad \text{and} \quad \forall^d.$$

$$\mathfrak{A} \models \exists^d R \phi(R)$$

iff

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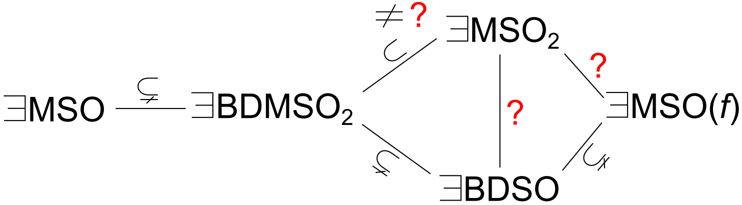
BDSO, MSO and MSO_2

- MSO: Quantify sets of vertices.
- MSO_2: Quantify sets of edges.
- $BDMSO_2$: Quantify sets of edges of bounded degree.
- BDSO: Quantify relations of bounded degree.
- MSO(f): Quantify unary functions.

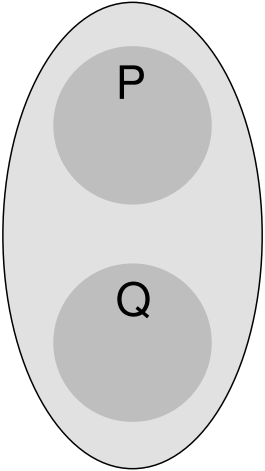
BDSO, MSO and MSO_2



Separating $\exists$BDMSO$_2$ and $\exists$MSO$_2$

- A query expressible in $\exists$MSO$_2$ but not in $\exists$BDMSO$_2$
- $S = \{E, P, Q\}$
- Surjective Homomorphism
- Is there a surjective homomorphism from the subgraph induced by P to the subgraph induced by Q?
- Internal Surjective Homomorphism
- Is there a subset of edges that forms surjective homomorphism from the subgraph induced by P to the subgraph induced by Q?

Separating $\exists BDMSO_2$ and $\exists MSO_2$



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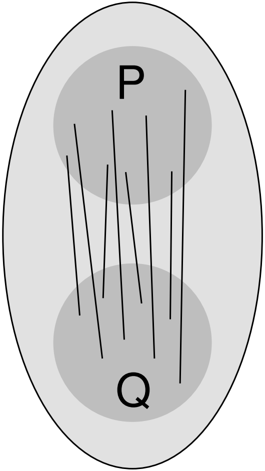

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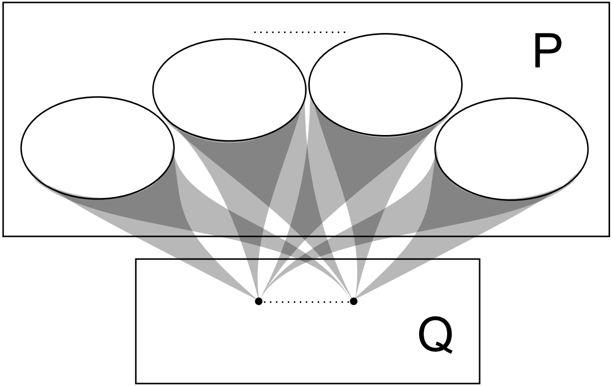
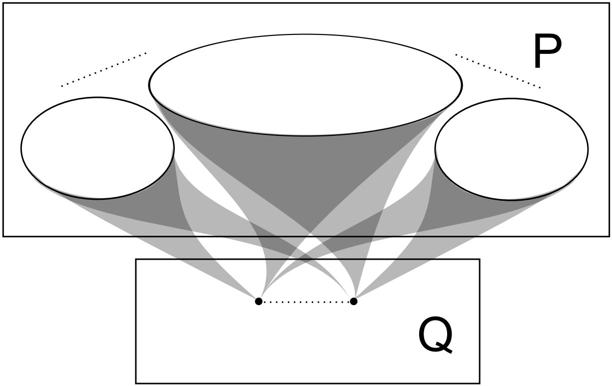


Separating $\exists$BDMSO$_2$ and $\exists$MSO$_2$

- The structure $\mathfrak{A}_{d,l,k}$ has $m >> max\{d, l, k\}$ cycles
- Q is a independent set of size m
- Each cycle has length $c >> m$
- The structure $\mathfrak{B}$ is obtained from $\mathfrak{A}_{d,l,k}$ by glueing two cycles
- $\mathfrak{B}_{d,l,k}$ has $m - 1$ cycles and, hence, there is no internal surjective homomorphism

Separating $\exists$BDMSO$_2$ and $\exists$MSO$_2$



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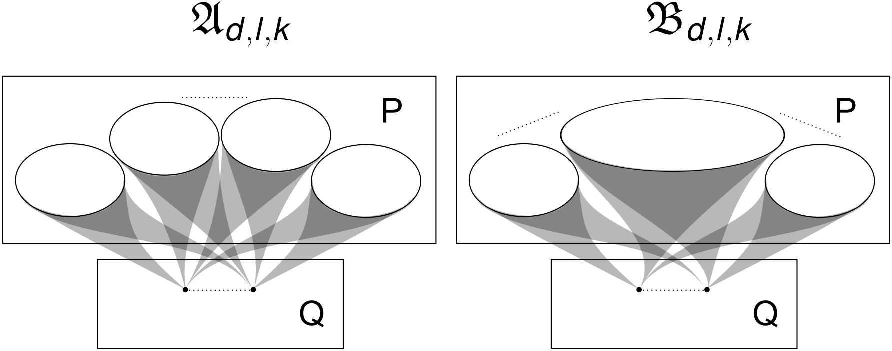
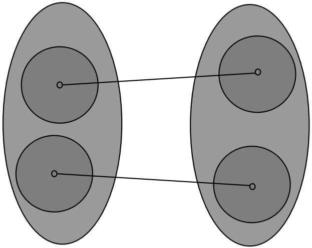


Separating $\exists$BDMSO$_2$ and $\exists$MSO$_2$

$$(\mathfrak{A}_{d,l,k}, R_1, \ldots, R_l) \leftrightarrow_{f(l)} (\mathfrak{B}_{d,l,k}, R'_1, \ldots, R'_l)$$

There is a bijection from $(\mathfrak{A}_{d,l,k}, R_1, \ldots, R_l)$ to $(\mathfrak{B}_{d,l,k}, R'_1, \ldots, R'_l)$ which preserves neighborhoods.



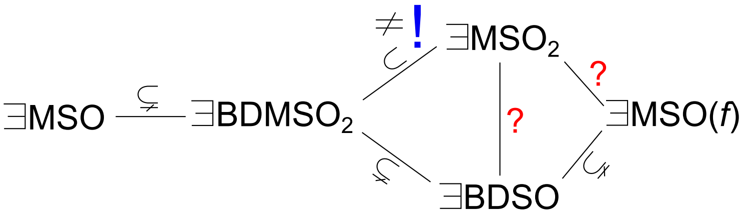
Separating $\exists$BDMSO$_2$ and $\exists$MSO$_2$

- Internal surjective homomorphism is not expressible by $\exists$BDMSO$_2$
- But internal surjective homomorphism is expressible in $\exists$MSO$_2$

  "Exists a set of edges between *P* and *Q* which is surjective

  and satisfies the homomorphism clauses"

BDSO, MSO and MSO_2



Future

- Separate the other fragments.
- Investigate the hierarchies.
- Parameterized complexity.

Thank You!