What Makes for a Good Paraconsistent Negation?

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What properties does one expect from negation?

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Pick your choice:

- reductio ad absurdum

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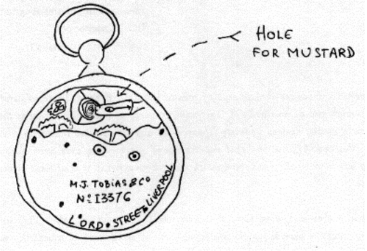

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Use your favorite recipe for producing *mustard watches*. [y.-j. ringard, 1990]

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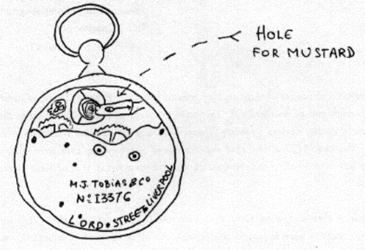

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Is there some minimal basis for agreement?!

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Boolean binary operators in 'binary print': [Béziau, 1996]

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	$\circledcirc_2^3$
T	*T*
T	*F*
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	$\circledcirc_2^2$
T	*T*
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	$\circledcirc_2^1$
T	*T*
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kinds of affirmation

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T	*F*
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	$\circledcirc^2_1$
T	*F*
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An *inessentialist* abstract approach: [JM, ON:plr 2005]

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(falsificatio) $\exists\varphi.\ \varphi \not\Vdash \sim\varphi$

(verificatio) $\exists\varphi.\ \sim\varphi \not\Vdash \varphi$

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An *inessentialist* abstract approach (generalized): [JM, ON:plr 2005]

(m-falsificatio) $\exists\varphi.\ \sim^m\varphi \nVdash \sim^{m+1}\varphi$

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Some after-effects:

$\exists\varphi.\ \nVdash {\sim}\varphi$

$\exists\varphi.\ {\sim}\varphi \nVdash$

$\exists\varphi.\ \varphi \not\dashv\vdash {\sim}\varphi$

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> Given any consistent logic $\mathcal{L}$, one can always find an inconsistent logic $\mathcal{IL}$ such that:
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Then, using the same trick as before:

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The **Paradox of Ineffable Inconsistencies**: [JM, II 2006]

Even though $\mathcal{IL}$ is $\sim$-inconsistent, it still respects *ex contradictione*:

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Lesson to be learned: A **decent** $\sim$-paraconsistent logic should not only have a $\sim$-inconsistent model, but a *non-dadaistic* such model.

On synonymity

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Let $\Phi, \Psi \subseteq \mathit{FRML}$. We say that *$\Phi$ is equivalent to Ψ in $\mathcal{L}$* if

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[Popper 1943, 1959, 1963]
I thought indeed about a system in which contradictory sentences were not 'embracing', that is, did not explode, but abandoned this system because it turned out to be *too weak*.

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Given a sentential context C, say that $\phi, \psi \in$ *FRML* are *C-equivalent in* $\mathcal{L}$ if both $C(\phi) \Vdash C(\psi)$ and $C(\psi) \Vdash C(\phi)$.

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We say that ϕ and ψ are *synonymous in $\mathcal{L}$* if they are C-equivalent with respect to any context C in *FRML*.

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Clearly, ϕ and ψ are atom-equivalent in $\mathcal{L}$ exactly when $\{\phi\}$ and $\{\psi\}$ are $\mathcal{L}$-equivalent.

We say that ϕ and ψ are *synonymous in $\mathcal{L}$* if they are C-equivalent with respect to any context C in $FRML$.

An n-ary connective © is called *congruential in $\mathcal{L}$* if it treats atom-equivalent arguments as synonymous.

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Note that:
(1) congruentiality $\prec$ extensionality $\prec$ determinedness (truth-functionality)
(2) typical examples of congruential logics: classical modal logics

On paraconsistency and congruentiality

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Can a paraconsistent logic be congruential?

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False start #1: logic $D2$ [Jaśkowski 1948–49]

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$\Gamma \Vdash_{D2} \alpha$ iff $J[\Gamma] \Vdash_{S5} J(\alpha)$, where $J = jask \circ wski$ and

$wski(p) = p$

$wski(\sim\alpha) = \sim wski(\alpha)$

$wski(\alpha \vee \beta) = wski(\alpha) \vee wski(\beta)$ $\qquad jask(\varphi) = \Diamond\varphi$

$wski(\alpha \wedge \beta) = wski(\alpha) \wedge \Diamond wski(\beta)$

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Despite appearances, $D2$ is not a modal logic! [JM, M&P 2005]

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Let $\neg\alpha := \alpha \supset \sim(\alpha \vee \sim\alpha)$.

Check that p and $\neg\neg p$ are atom-equivalent yet not $\sim$-equivalent in $D2$.

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Check that p and $\neg\neg p$ are atom-equivalent yet not $\sim$-equivalent in *D*2.

More generally, every $\sim$-paraconsistent extension of CPL^+ *fails congruentiality* if it happens to sanction the inference $\sim(\alpha \supset \beta) \Vdash \alpha \wedge \sim\beta$.

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Can a paraconsistent logic be congruential?

Say that a logic $\mathcal{L}$ is *C-decreasing* if $\phi \Vdash \psi$ implies $C(\psi) \Vdash C(\phi)$.

(1) typical examples of C-decreasing contexts: negative modalities in normal modal logics

(2) C-decreasing implies C-congruential

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False start #2: the status of contraposition

No (boldly) paraconsistent logic with a deductive implication $\supset$ can sanction the inference $\beta \supset \alpha \Vdash {\sim}\alpha \supset {\sim}\beta$. [Popper 1963]

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Indeed:

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Consider Kripke models for CPL^+ and add a unary connective $\smile$ such that:

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$$\begin{array}{lcl} \bot & := & \smile(p \supset p) \\ \neg\varphi & := & \varphi \supset \bot \\ \Box\varphi & := & \neg\smile\varphi \\ \Diamond\varphi & := & \smile\neg\varphi \\ \frown\varphi & := & \Box\neg\varphi \\ \circ\varphi & := & \varphi \supset \Box\varphi \end{array}$$

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Then the logics thereby defined. . . [JM, Nearly 2005]

. . . are $\smile$-decreasing, $\neg$-decreasing and $\frown$-decreasing
. . . are congruential
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The Challenge:

To do this without the help of an implication in the language!

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Axiomatizing normal (paraconsistent) modal logics on $FRML_{\wedge,\vee,\supset,\smile}$

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System $\mathcal{K}$: *CPL*$^{+}$ extended by

[K] $\vdash \smile(\alpha \wedge \beta) \supset (\smile\alpha \vee \smile\beta)$

[N1] If $\vdash \alpha \supset \beta$, then $\vdash \smile\beta \supset \smile\alpha$

[N2] If $\vdash \alpha$, then $\vdash \smile\alpha \supset \beta$

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System	*Axiom*	*Frames*
$\mathcal{KT}$	$\alpha \vee \smile\alpha$	reflexive
$\mathcal{KB}$	$\smile\smile\alpha \supset \alpha$	symmetric
$\mathcal{K}5$	$(\smile\alpha \wedge \smile\smile\alpha) \supset \beta$	euclidean
$\mathcal{K}4$	$(\frown\alpha \wedge \smile\frown\alpha) \supset \beta$	transitive
$\mathcal{K}2$	$(\smile\alpha \wedge \frown\smile\alpha) \supset \beta$	dense
$\mathcal{K}G$	$\smile\smile\alpha \supset \frown\frown\alpha$	confluent

Just imagine the possibilities

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Structures for studying negative modalities, and their companions:

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$$\frac{A \Rightarrow \varphi, B}{\frown[B], \smile\varphi \Rightarrow \smile[A]}\ (\smile\frown) \qquad \frac{A, \varphi \Rightarrow B}{\frown[B] \Rightarrow \frown\varphi, \smile[A]}\ (\frown\smile)$$

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A Derivability Adjustment Theorem: [A. Dodó & JM, NegMod 2014]

Let $\Pi^{\#}_{\neg}$ be the result of uniformly substituting each occurrence of the symbol $\neg$ in each sentence of Π by an occurrence of a unary symbol $\# \in \{\frown, \smile\}$. Then, inferences from *CL* may be recovered from $\mathcal{T}^{\mathrm{n}}$ in the following way:

$\Gamma^{\#}_{\neg} \vdash_{\mathrm{cl}} \Delta^{\#}_{\neg}$ iff there are finite sets $\Sigma_c, \Sigma_d \subseteq FRML$ such that $\circledsmile[\Sigma_c], \Gamma \vdash^{\mathcal{T}}_{\mathrm{n}} \Delta, \circledfrown[\Sigma_d]$

Furthermore, Σ_c may be constrained above to a finite collection of sub-sentences of Γ, and Σ_d may be constrained to a finite collection of sub-sentences of Δ.