Aplicações de Lógica em Complexidade Parametrizada

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O Problema Subset-SUM

Dado um conjunto de inteiros $X = \{x_1, \ldots, x_n\}$ *e um um inteiro* s, *existeum subconjunto de* X *cuja soma dos valores dê* s*?*

$$X = \{3,\ 4,\ 9,\ 11,\ 17,\ 25,\ 42,\ 45\}$$

$$\sum_{i \in S \subseteq [n]} x_i = 57?$$

O Problema Subset-SUM

$$X = \{\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8\ \}$$

$$X = \{\ 3,\ 4,\ 9,\ 11,\ 17,\ 25,\ 42,\ 45\ \}$$

$$\sum_{i \in S \subseteq [8]} x_i = 57 \ ?$$

O Subset-SUM parametrizado

O problema (k, M)-*SUM* é determinar, dado n inteiros

$$x_1, \ldots, x_n \in [0, M]$$

e um inteiro

$$s \in [0, M],$$

se existe um subconjunto $S \subseteq [n]$ com $|S| = k$ talque

$$\sum_{i \in S} x_i = s.$$

O problema k-*SUM* é definido como $(k, n^{f(k)})$-SUM

Parameterized Complexity Theory (1/5)

Um problema parametrizado é um par (P, χ) consistindo de um conjunto $P \subseteq \Sigma^*$ de strings sob Σ, e uma função χ computável em tempo polinômial tal que para todo $\omega \in P,\ \chi(\omega) = k$.

$$k\text{-SUM} := \{ \langle k \rangle \langle x_1 \rangle \ \ldots \ \langle x_n \rangle \langle s \rangle \in \{0,1\}^* \mid \text{ existe } S \subseteq [n], \text{ com } |S| = k \text{ tal que } \sum_{i \in S} x_i = s\}$$

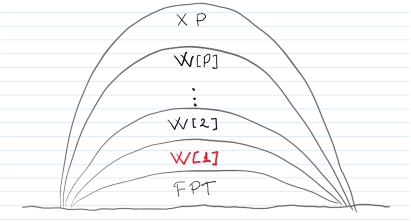
$$|w| = f(k) \cdot (n+1) \cdot \log n + \log k$$

Parameterized Complexity Theory (2/5)

Um problema parametrizado (P, χ) is fixed-parameter tractable se pode ser decidido por um algoritmo em

$$f(k) \cdot |\omega|^{O(1)},$$

para alguma função computável $f : \mathbb{N} \to \mathbb{N}$.



$$FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots W[P] \subseteq XP$$

Parameterized Complexity Theory (3/5)

p-MC(Φ)	
Instância:	Uma estrutura $\mathcal{A}$ e uma fórmula $\varphi \in \Phi$.
Parameter:	$\lvert\varphi\rvert$.
Problema:	Decidir se $\mathcal{A} \models \varphi$.

$\Sigma_{t,u}$

São fórmulas na forma

$$\exists x_{11} \ldots \exists x_{1m_1} \forall x_{21} \ldots \exists x_{2m_2} \ldots Q x_{t1} \ldots Q x_{t,m_t} \psi,$$

onde ψ é uma fórmula livre de quantificadores e $m_2, m_3, \ldots, m_t \leq u$.

Parameterized Complexity Theory (4/5)

Theorem

Seja $t \geq 1$. então, para todo $u \geq 1$,

$$p\text{-}MC(\Sigma_{t,u}) \text{ é completo para } W[t] \text{ sob reduções fpt.}.$$

Exemplo:

$$W[1] = [p\text{-MC}(\Sigma_1)]^{FPT}$$

Parameterized Complexity Theory (5/5)

k-CLIQUE:

$$clique_k := \exists x_1 \dots \exists x_k (\bigwedge_{1 \le i < j \le k} x_i \neq x_j \wedge \bigwedge_{1 \le i < j \le k} E x_i x_j).$$

k-CONJUNTO-INDEPENDENTE:

$$is_k := \exists x_1 \dots \exists x_k (\bigwedge_{1 \le i < j \le k} x_i \neq x_j \wedge \bigwedge_{1 \le i < j \le k} \neg E x_i x_j).$$

k-CONJUNTO-DOMINANTE:

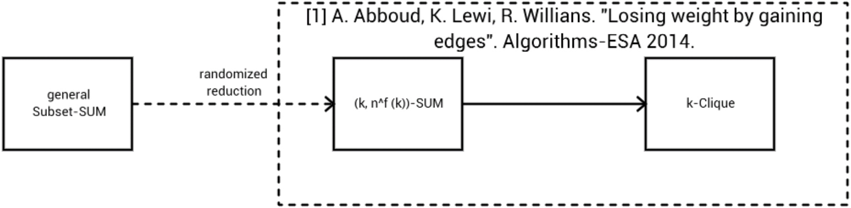
$$ds_k := \exists x_1 \dots \exists x_k (\bigwedge_{1 \le i < j \le k} x_i \neq x_j \wedge \forall y \bigwedge_{1 \le i < j \le k} (E x_i y \vee x_i = y)).$$

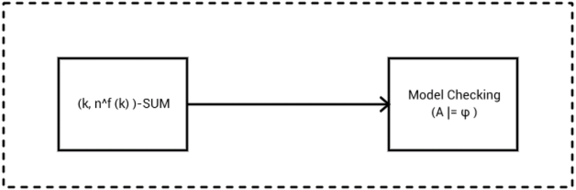
k-SUM é W[1]-completo

i Downey and Fellows (1992,1995) provaram que o k-SUM é $W[1]$-hard e em $W[P]$.

ii Buss and Islam (2007) provaram que k-SUM está em $W[3]$.

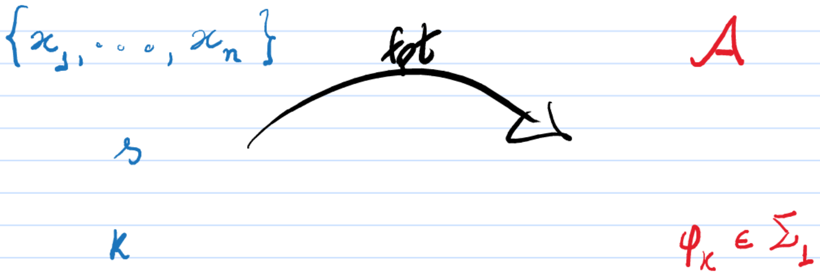
iii Abboud, Lewi and Williams (2014) provaram k-SUM está em $W[1]$, logo, $W[1]$-completo.


general
Subset-SUM
randomized
reduction
[1] A. Abboud, K. Lewi, R. Willians. "Losing weight by gaining edges". Algorithms-ESA 2014.
(k, n^f (k))-SUM
k-Clique


(k, n^f (k))-SUM
Model Checking
(A |= φ)

k-SUM é W[1]-completo

Precisamos codificar a entrada para o problema de model checking: Uma estrutura $\mathcal{A}$ *e uma fórmula* $\varphi_k \in \Sigma_1$ *such that* $\mathcal{A} \models \varphi_k$



Tentativa: $\varphi_k \in \Sigma_1$

k-SUM pode ser expresso por fórmulas $\{\varphi_k\}$ em Σ_1 na forma:

$$\exists u_1, \ldots, u_k, v_1, \ldots, v_k$$

$$\bigwedge_{1 \leq i,j \leq k} ((u_i \neq u_j) \wedge (v_i \neq v_j)) \wedge \bigwedge_{i=1}^{k} R(u_i, v_i) \wedge \left(\sum^{k} v_i = s \right).$$

$$\left(\sum^{k} v_i = s \right) \equiv \exists r_1 r_2 \ldots r_{k-2} \quad PLUS^3(v_1, r_1, s)$$

$$\ldots \wedge \bigwedge_{i=2}^{k-2} PLUS^3(v_i, r_i, r_{i-1}) \wedge PLUS^3(v_{k-1}, v_k, r_{k-2}).$$

O vocabulário

$\tau := \{\mathsf{R}^{f(k)+1},\ =,\ \leq,\ \mathsf{PLUS}^3, \mathsf{Succ},\ 0,\ 1,\ n+1,\ s_1\ \dots\ s_{f(k)}\ \}$

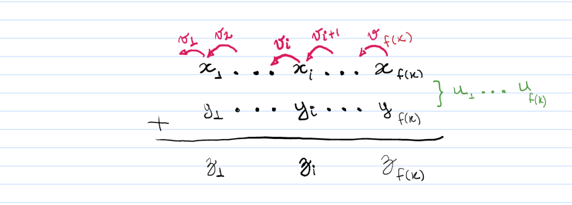
Uma relação R $(f(k)+1)$-ária

$R = \{(i, \overline{x}_i) \mid \langle x_{i1} \dots x_{if(k)} \rangle_{n+1} \in X\}$

$X = \{3,\ 4,\ 9,\ 11,\ 17,\ 25,\ 42,\ 45\}$

$R = \{(1, \langle 3 \rangle),\ (2, \langle 4 \rangle),\ (3, \langle 9 \rangle),\ (4, \langle 11 \rangle),\ (5, \langle 17 \rangle),\ (6, \langle 25 \rangle),\ (7, \langle 42 \rangle),\ (8, \langle 45 \rangle)\}$

$\mathsf{PLUS}^{3f(k)} \in \Sigma_1$



$\varphi_k \in \Sigma_1$ expressing that k-SUM problem has a solution

k-SUM can be expressed by $\{\varphi_k\}$ formulas in Σ_1 in the form:

$$(\exists u_1, \ldots, u_k, \overline{v}_1, \ldots, \overline{v}_{f(k)})$$

$$\bigwedge_{1 \leq i,j \leq k} ((u_i \neq u_j) \wedge (\overline{v}_i \neq \overline{v}_j)) \wedge \bigwedge_{i=1}^{k} R(u_i, \overline{v}_i) \wedge \left(\sum^{k} \overline{v}_i = \overline{s} \right),$$

where $\overline{s} := \langle s_1, \ldots s_{f(k)} \rangle$.

$$(\exists \overline{r}_1 \overline{r}_2 \ldots \overline{r}_{f(k)-2}) \quad PLUS^{3f(k)}(\overline{v}_1, \overline{r}_1, \overline{s})$$

$$\wedge \bigwedge_{i=2}^{f(k)-2} PLUS^{3f(k)}(\overline{v}_i, \overline{r}_i, \overline{r}_{i-1}) \wedge PLUS^{3f(k)}(\overline{v}_{k-1}, \overline{v}_k, \overline{r}_{f(k)-2}).$$

$$k\text{-SUM} := \{\, w \quad | \quad w = \langle k \rangle \ \langle x_1 \rangle \ \ldots \ \langle x_n \rangle \ \langle s \rangle \in \{0,1\}^* \ ; \text{ there exists } S \subseteq [n] \text{ with } |S| = k \text{ such that } \sum_{i \in S} x_i = s \}$$

$$|w| = f(k) \cdot (n+1) \cdot \log n + \log k$$

The structure's encoding:

$$\langle \mathcal{A} \rangle \ := \underbrace{\langle 0 \rangle \ldots \langle n \rangle \langle \langle R \rangle \langle \langle 1 \rangle \langle x_1 \rangle \rangle \ldots \langle \langle n \rangle \langle x_n \rangle \rangle \rangle \langle 0 \rangle \ldots \langle n+1 \rangle \langle s_1 \rangle \ldots \langle s_{f(k)} \rangle}_{|\langle \mathcal{A} \rangle| \in O(n \cdot f(k) \cdot \log n)}$$

Abboud, Amir and Lewi, Kevin and Williams, Ryan, *Losing Weight by Gaining Edges*, **Algorithms - ESA 2014**, (AndreasS Schulz and Dorothea Wagner, ed.), Springer, Berlin Heidelberg, 2014, pp. 1–12.

Downey, Rodney G and Fellows, Michael R and Regan, K, *Descriptive Complexity and the W Hierarchy*, **Proof Complexity and Feasible Arithmetics: DIMACS Workshop, April 21-24, 1996**, (P. Beame and S. Buss, ed.), vol. 39, 1998, pp. 119–134.

Jörg Flum and Martin Grohe, *Fixed-parameter tractability, definability, and model checking*, **SIAM Journal on Computing**, vol. 31 (2001), no. 1,pp. 113-145.

$$\mathsf{PLUS}^{3f(k)}(x_1, \ldots, x_{f(k)}, y_1, \ldots, y_{f(k)}, z_1, \ldots z_{f(k)}) :=$$

$$\exists u_1 \ldots u_{f(k)}\ v_1 \ldots v_{f(k)}$$

$$(\mathsf{PLUS}(x_{f(k)}, y_{f(k)}, z_{f(k)}) \land (v_{f(k)} = 0) \land (z_{f(k)} \neq n+1))$$
$$\lor (\mathsf{PLUS}(x_{f(k)}, u_{f(k)}, n+1) \land \mathsf{PLUS}(u_{f(k)}, z_{f(k)}, y_{f(k)}) \land (v_{f(k)} = 1))$$
$$\lor (\mathsf{PLUS}(u_{f(k)}, y_{f(k)}, n+1) \land \mathsf{PLUS}(u_{f(k)}, z_{f(k)}, x_{f(k)}) \land (v_{f(k)} = 1))$$

$$\bigwedge_{i=f(k)-1}^{1} (\ ((v_{i+1} = 0) \to (\mathsf{PLUS}(x_i, y_i, z_i) \land (v_i = 0) \land (z_i \neq n+1))$$
$$\lor (\mathsf{PLUS}(x_i, u_i, n+1) \land \mathsf{PLUS}(u_i, z_i, y_i) \land (v_i = 1))$$
$$\lor (\mathsf{PLUS}(u_i, y_i, n+1) \land \mathsf{PLUS}(u_i, z_i, x_i) \land (v_i = 1)))$$
$$\land ((v_{i+1} = 1) \to (\mathsf{PLUS}(x_i, y_i, z_i) \land (v_i = 0) \land (z_i \neq n+1))$$
$$\lor (\mathsf{PLUS}(x_i, u_i, n+1) \land \mathsf{PLUS}(u_i, z_i, y_i) \land (v_i = 1))$$
$$\lor (\mathsf{PLUS}(u_i, y_i, n+1) \land \mathsf{PLUS}(u_i, z_i, x_i) \land (v_i = 1))))$$

A parameterized problem (P, χ) is said to be XP, if there exists an algorithm that decides P in

$$|\omega|^{\mathcal{O}(f(k))},$$

for some comptable function $f : \mathbb{N} \to \mathbb{N}$.