Chapter 1

THEORETICAL AND COMPUTATIONAL RESULTS FOR A LINEAR BILEVEL PROBLEM

Manoel Campêlo

Dep. Estatística e Mat. Aplicada, Universidade Federal do Ceará - UFC

mcampelo@ufc.br

Susana Scheimberg

DCC/IM, COPPE/Sistemas, Universidade Federal do Rio de Janeiro - UFRJ

susana@cos.ufrj.br

Abstract In this paper a linear bilevel programming problem (LBP) is considered. Two related problems (P) and $P(M)$ are examined. The three problems are shown to be simultaneously infeasible or unbounded for all $M \geq 0$ or else they have equivalent global solution sets for some finite $M^\star$ and all $M > M^\star$. The concept of equilibrium point of $P(M)$ is introduced and its properties are analysed. An algorithm to obtain an ε-solution to (LBP) is developed. It is based on equilibrium point properties and a modified version of outer approximation. Infeasible and unbounded cases are also identified by the method. Computational results are given.

Keywords: Bilevel linear programming, global optimization, exact penalty methods.

1. INTRODUCTION

In this article we focus our attention on the following linear bilevel program:

$$
\begin{array}{lll}
\text{(LBP)} & \max_{x,y} & c_1^T x + c_2^T y \\
 & \text{s.t.} & x \geq 0, y \text{ solves} \\
 & & \quad \begin{array}{ll} \max_y & b^T y \\ \text{s.t.} & A_1 x + A_2 y \leq a \\ & y \geq 0 \end{array}
\end{array}
$$

where $c_1, x \in \mathcal{R}^{n_1}$, $c_2, b, y \in \mathcal{R}^{n_2}$, $a \in \mathcal{R}^m$, $A_1 \in \mathcal{R}^{m \times n_1}$ and $A_2 \in \mathcal{R}^{m \times n_2}$. Bilevel Programming has applicability in many domains, e.g. in network design, transportation, game theory, engineering and economics [13].

Let us note that (LBP) is a nonconvex problem. Actually it is strongly NP-hard [8]. This problem has been extensively studied in the literature, see e.g. [1, 2, 11, 17]. We refer to [16] for a bibliography survey.

The aim of this paper is to develop an algorithm for finding an ε-global solution to problem (LBP). It is based on the new notion of equilibrium point and a modified version of the outer approximation method. In the procedure we can identify two parts. The first one achieves a local solution of a problem related to (LBP). The second part introduces a cut that separates the feasible set containing the global solution from those points just obtained. Thus, we consider problem (LBP) with a linear constraint in the first level. This problem is denoted by (LBP1). We observe that the method recognizes infeasible and unbounded cases.

The paper is organized as follows. Section 2 contains theoretical background supporting the proposed method. In section 3 we present the algorithm. We prove that it is well defined and finishes in a finite number of steps. Finally, we give some computational results.

2. THEORETICAL PRELIMINARIES

2.1 AUXILIARY PROBLEMS

In this subsection we define two auxiliary problems of (LBP). They will be the basis of our approach. We also introduce a simplified notation. Finally we give a property showing a way to find feasible points of (LBP) and related problems, which is computationally attractive.

The first auxiliary problem (P) is obtained by replacing the inner linear problem of (LBP) by its KKT conditions. The second one $P(M)$ comes from a penalization of the complementarity constraints in (P) with a parameter $M \geq 0$. Thus, we have the following models:

$$
\begin{array}{lll}
(P) & & \\
\max & c_1^T x + c_2^T y & \\
\text{s.t.} & A_1 x + A_2 y + w = a, & \\
& x \geq 0, y \geq 0, w \geq 0, & \\
& A_2^T u - v = b, & \\
& u \geq 0, v \geq 0, & \\
& u^T w + v^T y = 0 &
\end{array}
\qquad
\begin{array}{ll}
P(M) & \\
\max & c_1^T x + c_2^T y - M(u^T w + v^T y) \\
\text{s.t.} & A_1 x + A_2 y + w = a, \\
& x \geq 0, y \geq 0, w \geq 0, \\
& A_2^T u - v = b, \\
& u \geq 0, v \geq 0
\end{array}
$$

where $w \in \mathcal{R}^m$, $u \in \mathcal{R}^m$ and $v \in \mathcal{R}^{n_2}$.

Formulation (P) is often used instead of model (LBP), see e.g. [3, 11, 14, 17]. Indeed, they are globally equivalent, in the sense that a global solution of (P), if it exists, corresponds to a global solution of (LBP) and vice-versa. The correspondence also holds in the cases of infeasibility and unboundedness.

However, a local solution of (P) may not be associated with a local solution of (LBP). In [7] we give a counterexample.

The penalty function has also been considered in the literature [1, 14, 17] as an approach to identify global optima of (LBP) under restrictive assumptions. Actually, under compactness hypothesis on the set defined by the primal constraints, Bard [3] shows that extreme global solutions of $P(M^\star)$ yield extreme global optima of (P) and vice-versa, for some finite $M^\star \geq 0$. In [6] we prove that it is sufficient to consider the objective function of (P) bounded above on that set. Now, we show that this penalty is still useful in a wider context involving unbounded and infeasible situations. We point out that the penalty parameter can be implicitly considered, which is computationally attractive.

In order to simplify the notation, we consider the block matrices $A = [A_1\, A_2\, I_m] \in \mathcal{R}^{m\times n}$, $B = [0 - I_{n_2}\, A_2^T] \in \mathcal{R}^{n_2\times n}$ and $c^T = (c_1^T, c_2^T, 0) \in \mathcal{R}^n$, where $n = n_1 + n_2 + m$, I_k is the $(k \times k)$-identity matrix and 0 is a null matrix of appropriate dimension for each case. We define the sets $Z = \{z \in \mathcal{R}^n : Az = a, z^T = (x^T, y^T, w^T) \geq 0\}$ and $S = \{s \in \mathcal{R}^n : Bs = b, s^T = (0, v^T, u^T) \geq 0\}$. Thus, the auxiliary problems are rewritten as:

$$
(P)\quad \begin{array}{ll} \max & F(z,s) = c^T z \\ \text{s.t.} & z \in Z, s \in S \\ & s^T z = 0 \end{array} \qquad P(M)\quad \begin{array}{ll} \max & F_M(z,s) = c^T z - M s^T z \\ \text{s.t.} & z \in Z, s \in S \end{array}
$$

We denote by X_v the vertex set of a polyhedron X and by $V[\mathcal{P}]$ the optimal value of a maximization problem $(\mathcal{P})$, regarding that $V[\mathcal{P}]$ is $-\infty$ or $+\infty$ if $(\mathcal{P})$ is infeasible or unbounded respectively. Let us recall that the infinity norm of $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is $\| \nu \|_\infty = \max\{|\nu_i| : 1 \leq i \leq p\}$ and $B_\varepsilon(\bar\nu) = \{\nu \in \mathcal{R}^p : \| \nu - \bar\nu \|_\infty \leq \varepsilon\}$ is an ε-neighborhood of $\bar\nu \in \mathcal{R}^p$.

Proposition 1 *If Z and S are nonempty sets then (P) is feasible. Moreover, for each $z \in Z$ and $s \in S$, it holds:*

(1) If $\hat s \in \arg\min\{z^T s : s \in S_v\}$ and $\hat z \in \arg\min\{\hat s^T z : s \in Z\}$ then $(\hat z, \hat s)$ is feasible to (P).

(2) If $\hat z \in \arg\min\{s^T z : z \in Z_v\}$ and $\hat s \in \arg\min\{\hat z^T s : s \in S\}$ then $(\hat z, \hat s)$ is feasible to (P).

Proof: **(1)** Let $z \in Z$ and $\hat s \in \arg\min\{z^T s : s \in S_v\}$. Consider $B = [E\, R]$ where E is a basis of B associated to $\hat s^T = ((E^{-1}b)^T, 0)$. Let be $z^T = (z_E^T, z_R^T)$. Consider the optimal reduced cost vetor $\tilde z^T = (0, z_R^T - z_E^T E^{-1} R) = z^T - z_E^T E^{-1} B \geq 0$. Since $BA^T = 0$ we obtain that $A\tilde z = Az = a$. Hence, $\tilde z \in Z$. The partitions of $\tilde z$ and $\hat s$ results $\hat s^T \tilde z = 0$. So it must be $\hat s^T \hat z = 0$ for each $\hat z \in \arg\min\{\hat s^T z : z \in Z\}$.
(2) It is proved in a similar way. ■

2.2 GLOBAL EQUIVALENCE

Here we characterize the existence or nonexistence of a global solution to problem (LBP) by considering the auxiliary problems. We start with the following property:

Proposition 2 *If problem (P) is feasible and problem $P(M_0)$ has solution for some $M_0 \geq 0$, then there exists $M^\star \geq M_0$ such that:*

(1) There exists $(z^\star, s^\star) \in Z_v \times S_v$ satisfying $(s^\star)^T z^\star = 0$ and $V[P] = V[P(M)] = F(z^\star, s^\star)$ for all $M \geq M^\star$.

(2) Moreover, (P) and $P(M)$ have the same nonempty global solution set for all $M > M^\star$.

Proof: **(1)** As $P(M)$ is less restricted than (P) and $s^T z \geq 0$ for all $(z,s) \in Z \times S$, $P(M)$ has solution for all $M \geq M_0$. Moreover, at least one of its solutions is attained at a vertex, since it is a bilinear program [10, proposition IX.1]. Then, it follows that

$$V[P(M)] = \max\{F(z,s) - Ms^T z : z \in Z_v, s \in S_v\} \quad \forall M \geq M_0. \quad (1.1)$$

Let us split $Z_v \times S_v$ into the finite subsets, $C_0 = \{(z,s) \in Z_v \times S_v : z^T s = 0\}$ and $C_1 = (Z_v \times S_v) \setminus C_0$. As (P) is feasible, it must be $C_0 \neq \emptyset$. Let $(z^\star, s^\star) \in C_0$ satisfying $F(z^\star, s^\star) = \max\{F(z,s) : (z,s) \in C_0\}$. If $C_1 \neq \emptyset$, let us take

$$M^\star = \max\left\{M_0,\ \max\{(F(z,s) - F(z^\star, s^\star))/s^T z : (z,s) \in C_1\}\right\}. \quad (1.2)$$

Thus, for all $M \geq M^\star$ we have

$$L(M) = \max\{F(z,s) - Ms^T z : (z,s) \in C_1\} \leq F(z^\star, s^\star) \quad (1.3)$$

By (1.1)-(1.3) and taking into account that $M^\star \geq M_0$, it follows that

$$V[P(M)] = \max\{F(z^\star, s^\star), L(M)\} = F(z^\star, s^\star) \quad \forall M \geq M^\star. \quad (1.4)$$

If $C_1 = \emptyset$, we trivially have $V[P(M)] = F(z^\star, s^\star)$ for all $M \geq M^\star = M_0$. So, it results

$$(s^\star)^T z^\star = 0 \quad \text{and} \quad V[P] = V[P(M)] = F(z^\star, s^\star) \quad \forall M \geq M^\star. \quad (1.5)$$

(2) Let $\bar{M} > M^\star$. Let us denote by $O(P)$ and $O(P(\bar{M}))$ the optimal sets of problems (P) and $P(\bar{M})$. By (1.5), we have that $O(P) \neq \emptyset$ and $O(P(\bar{M})) \neq \emptyset$, with $V[P] = V[P(\bar{M})]$. Moreover, since any feasible solution of (P) is feasible to $P(\bar{M})$, it follows that $O(P) \subseteq O(P(\bar{M}))$.

Conversely, let $(z,s) \in O(P(\bar{M}))$. Using again (1.5) together with $\bar{M} > M^\star$, we have that $V[P] = V[P(\bar{M})] = F(z,s) - \bar{M}s^T z \leq F(z,s) -$

$M^\star s^T z \le V[P(M^\star)] = V[P]$. Thus, equality holds among the relations. It yields that $(\bar{M} - M^\star)s^T z = 0$, which implies $s^T z = 0$. Therefore, (z, s) is feasible to (P) and $V[P] = F(z, s)$, that is, $(z, s) \in O(P)$. Hence, $O(P(\bar{M})) \subseteq O(P)$. So, we obtain the desired conclusion. ■

Let us note that the definition of $M^\star$ coincides with the expression given in [3], when $C_1 \ne \emptyset$ and $P(M)$ has solution for all M. Also, we observe that item (2) of proposition 2 may not hold for $M = M^\star$ [7]. Finally, it is worth stressing that the hypotheses of the above proposition are weaker than those ones considered in [3, 6].

Now we address our attention to the situations not covered by proposition 2, that is, unboundedness and infeasibility. For the first case, we are able to present the following equivalence:

Proposition 3 *Problem (P) is unbounded if, and only if, problem $P(M)$ is unbounded for all $M \ge 0$.*

Proof: If (P) is unbounded, $P(M)$ is unbounded for all $M \ge 0$, since $V[P(M)] \ge V[P] = +\infty$ for all $M \ge 0$. Now, in order to prove the converse claim we consider the recession cone of Z restricted to the unitary hypercube $Z_r = \{h \in \mathcal{R}^n : Ah = 0, h \ge 0, \| h \|_\infty \le 1\}$. Let $P(M)$ be unbounded for all $M \ge 0$. It means that there exists $(h^\star, s^\star) \in (Z_r)_v \times S_v$ such that $(s^\star)^T h^\star = 0$ and $c^T h^\star > 0$. In addition, since $Z \ne \emptyset$ and $(s^\star)^T z \ge 0$ for all $z \in Z$, we can take $z^\star \in \arg\min\{(s^\star)^T z : z \in Z_v\}$. We are going to prove that there exists $\hat{s} \in S$, such that $(z^\star + \alpha h^\star, \hat{s})$ is feasible to (P) for all $\alpha \ge 0$. Let $A = [D\ N]$ be a partition of A, where D is a basis associated with $(z^\star)^T = ((D^{-1}a)^T, 0)$. Let $(s^\star)^T = ((s^\star_D)^T, (s^\star_N)^T)$ and consider the optimal reduced cost vector $\hat{s}_o^T = \left(0, (s^\star_N)^T - (s^\star_D)^T D^{-1} N\right) = (s^\star)^T - (s^\star_D)^T D^{-1} A \ge 0$. By the partitions of $\hat{s}_o$ and $z^\star$, we have $\hat{s}_o^T z^\star = 0$. Moreover, as $Ah^\star = 0$ and $AB^T = 0$, the second expression for $\hat{s}_o$ yields $\hat{s}_o^T h^\star = (s^\star)^T h^\star = 0$ and $B\hat{s}_o = Bs^\star = b$. Now, let us partition $\hat{s}_o^T = (\hat{r}^T, \hat{v}^T, \hat{u}^T)$ according to the block matrix B and define $\hat{s}^T = (0, \hat{v}^T, \hat{u}^T)$. It is clear that $\hat{s} \ge 0$ and $B\hat{s} = B\hat{s}_o = b$ that is, $\hat{s} \in S$. Furthermore, since $\hat{s}_o \ge 0$, $z^\star \ge 0$ and $h^\star \ge 0$, we have $\hat{s}^T z^\star = \hat{s}_o^T z^\star = 0$ and $\hat{s}^T h^\star = \hat{s}_o^T h^\star = 0$. Thus, $(z^\star + \alpha h^\star, \hat{s})$ is feasible to (P) for all $\alpha \ge 0$. Since $c^T h^\star > 0$, it follows that problem (P) is unbounded. ■

For the infeasible case we have the next property:

Proposition 4 *Let $\bar{M} \ge 0$. Problem (P) is infeasible if, and only if, problem $P(\bar{M})$ is infeasible.*

Proof: Let $\bar{M} \ge 0$. If $P(\bar{M})$ is infeasible, clearly (P) is infeasible. Conversely, assume that $P(\bar{M})$ is feasible. Then, Z and S are nonempty. By proposition 1, (P) is feasible. ■

Now we summarize the relations between the auxiliary problems as follows:

Theorem 5 *Exactly one of the following assertions must hold:*

(1) problems (P) and $P(M)$ are infeasible for all $M \geq 0$;
(2) problems (P) and $P(M)$ are unbounded for all $M \geq 0$;
(3) problems (P) and $P(M)$ have the same nonempty global solution set for all $M > M^\star$ for some $M^\star \geq 0$.

Finally, we express theorem 5 in terms of the original problem:

Theorem 6 *Problems (LBP) and $P(M)$ are both infeasible or unbounded for all $M \geq 0$, or else they have solutions for some M_0. In this case, $z^\star$ is a global solution of (LBP) if, and only if, there is $s^\star$ such that $(z^\star, s^\star)$ is a global solution of $P(M)$ for all $M > M^\star$, for some $M^\star \geq M_0$.*

Remark: All the above results remain valid when we add to the first level linear constraints involving only variable x. Besides, even if these constraints depend on variable y, proposition 2 still holds and we can guarantee as well the last part of theorem 6. However, propositions 3 and 4 cannot be extended to this more general (LBP). See [7] for details.

2.3 EQUILIBRIUM POINTS

In this part we consider the linear bilevel problem with linear constraints in the first level, (LBP1). Let us denote its related problems by $(\tilde{P})$ and $\tilde{P}(M)$. We introduce the concept of equilibrium point of the parametric problem. We establish some properties used in the algorithmic section. We also characterize local solutions of $(\tilde{P})$ in terms of this new concept.

Problems $(\tilde{P})$ and $\tilde{P}(M)$ can be obtained from (P) and $P(M)$ by replacing Z by $\tilde{Z} = Z \cap \{z \in \mathcal{R}^n : D_1 x + D_2 y \leq d\}$, where $D_1 \in \mathcal{R}^{p \times n_1}$, $D_2 \in \mathcal{R}^{p \times n_2}$ and $d \in \mathcal{R}^p$. We introduce the following notion.

Definition 7 *A point $(\bar{z}, \bar{s}) \in \tilde{Z} \times S$ is an* **equilibrium point** *of the penalized problem if there is $\bar{M} \geq 0$ such that, for each $M \geq \bar{M}$, it holds*

$$\max\{F_M(\bar{z}, s) : s \in S\} = F_M(\bar{z}, \bar{s}) = \max\{F_M(z, \bar{s}) : z \in \tilde{Z}\}. \qquad (1.6)$$

The equilibrium required in the above definition extends that one considered by the mountain climbing algorithm given for a bilinear program [12]. Indeed, in our case, equality (1.6) must be fulfilled for all $M \geq \bar{M}$, which means that the equilibrium has to be held by a family of parametric bilinear problems $P(M)$. This yields the following property related to the complementarity term of the penalty function.

Lemma 8 *If $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem then*

$$\min\{\bar{z}^T s : s \in S\} = \bar{s}^T \bar{z} = \min\{\bar{s}^T z : z \in \tilde{Z}\}. \qquad (1.7)$$

Proof: Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem. The left equality in (1.7) results from the first one given in (1.6). Now, suppose, by contradiction, that $\bar{s}^T\bar{z} > \bar{s}^T\hat{z} = \min\{\bar{s}^Tz : z \in \tilde{Z}\}$. Then, $\bar{s}^T(\bar{z} - \hat{z}) > 0$. Take $M > \max\{\bar{M}, c^T(\bar{z}-\hat{z})/\bar{s}^T(\bar{z}-\hat{z})\}$. Thus, $c^T\hat{z} - M\bar{s}^T\hat{z} > c^T\bar{z} - M\bar{s}^T\bar{z}$, which contradicts the second equality in (1.6), since $M > \bar{M}$. Therefore, $\bar{s}^T\bar{z} = \min\{\bar{s}^Tz : z \in Z\}$, and the proof is complete. ∎

When $\tilde{Z} = Z$ we can show that an equilibrium point $(\bar{z}, \bar{s})$ satisfies $\bar{z}^T\bar{s} = 0$. This is essential to establish the equivalence between an equilibrium point and a local solution of (P) [7]. In the general case, we have the following characterization of a local solution of $(\tilde{P})$:

Theorem 9 *Let $(\bar{z}, \bar{s})$ be feasible to $(\tilde{P})$. The point $(\bar{z}, \bar{s})$ is a local solution of $(\tilde{P})$, if and only if, $(\bar{z}, \bar{s})$ is an equilibrium point of the corresponding penalized problem.*

Proof: Assume that $(\bar{z},\ \bar{s})$ is a local solution to $(\tilde{P})$. So, there is $\varepsilon > 0$ such that $c^T\bar{z} \geq c^Tz$ for all $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$. If $(\bar{z}, \bar{s})$ is not an equilibrium point, for all $\bar{M} \geq 0$ there are $M > \bar{M}$ and $z \in \tilde{Z}$ such that $F_M(z, \bar{s}) > F_M(\bar{z}, \bar{s}) = c^T\bar{z}$. Define $z_\alpha = \alpha z + (1 - \alpha)\bar{z}$ such that $\alpha \in (0, 1)$ and $(z_\alpha, \bar{s}) \in B_\varepsilon(\bar{z}, \bar{s})$. The convexity of $\tilde{Z}$ assures that $z_\alpha \in \tilde{Z}$. Hence, $c^Tz_\alpha \geq F_M(z_\alpha, \bar{s}) = \alpha F_M(z, \bar{s}) + (1 - \alpha)c^T\bar{z} > c^T\bar{z}$. This yields is a contradiction.

Conversely, let $(\bar{z}, \bar{s})$ be feasible to $(\tilde{P})$ and an equilibrium point of the penalized problem. Then, $\bar{z}^T\bar{s} = 0$ and there is $\bar{M} \geq 0$ such that

$$c^Tz - \bar{M}\bar{s}^Tz \leq c^T\bar{z} \quad \forall z \in \tilde{Z}. \tag{1.8}$$

If $\bar{s} = 0$, the inequality above implies that $c^Tz \leq c^T\bar{z}$ for all (z, s) feasible to $(\tilde{P})$. Now suppose that $\bar{s} \neq 0$. Let us split the index set $J = \{1, 2, \ldots, n\}$ into the subsets $J_0 = \{j \in J : \bar{s}_j = 0\}$ and $J_1 = J \setminus J_0$. Since $\bar{s} \neq 0$, it must be $J_1 \neq \emptyset$. For each $j \in J_1$, let us define the set $C_j = \{s \in \mathcal{R}^n : s_j = 0\}$. Then, set C_j is closed, convex and nonempty $(0 \in C_j)$. As $\bar{s} \notin C_j$, there is an hyperplane separating $\bar{s}$ and C_j strongly. So it is known that $d(\bar{s}, C_j) = \min\{\| s - \bar{s} \|_\infty : s \in C_j\} > 0$. Hence, there exists $\varepsilon \in \mathcal{R}$ such that $0 < \varepsilon < \min\{d(\bar{s}, C_j) : j \in J_1\}$. Let $s \in S \cap B_\varepsilon(\bar{s}) \neq \emptyset$. Then, $s \notin C_j$ for all $j \in J_1$, that is, $s_j > 0$ for all $j \in J_1$. Then,

$$0 = s^Tz = \bar{s}^Tz \quad \forall (z, s) \in B_\varepsilon(\bar{z}, \bar{s}),\ \text{feasible to } (\tilde{P}). \tag{1.9}$$

Using (1.8)-(1.9), we conclude that $c^Tz \leq c^T\bar{z}$ for all $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to $(\tilde{P})$. Therefore, $(\bar{z}, \bar{s})$ is a local solution of $(\tilde{P})$. ∎

3. ALGORITHMS

3.1 EQUILIBRIUM POINT ALGORITHM

In this subsection we describe an algorithm for finding an equilibrium point. Such a procedure will be used in the global algorithm.

Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem. According to definition 7, $\bar{z}$ and $\bar{s}$ are respectively solutions of the following parametric linear problems for all $M \geq \bar{M}$.

$$\begin{array}{lll} \tilde{P}(M;\bar{s}) & \max & F_M(z,\bar{s}) \\ & \text{s.t.} & z \in \tilde{Z} \end{array} \qquad\qquad \begin{array}{lll} \tilde{P}(M;\bar{z}) & \max & F_M(\bar{z},s) \\ & \text{s.t.} & s \in S \end{array}$$

We can solve problems $\tilde{P}(M;\bar{s})$ and $\tilde{P}(M;\bar{z})$ considering M implicitily as a large dominating value like in the big-M simplex method [4]. Particularly, the value of M does not affect the solution point of $\tilde{P}(M;\bar{z})$.

We describe below a procedure to find an equilibrium point starting from a point $z^0 \in \tilde{Z}$. A similar algorihtm can be designed from a point $s^0 \in S$. We assume *a priori* that $\tilde{Z} \times S \neq \emptyset$.

Algorithm 1 (Equilibrium Point Algorithm)

0. Let $z^0 \in \tilde{Z}$. Set $k = 0$.
1. Solve $\tilde{P}(M;z^k)$ using simplex method to obtain a solution s^k.
2. Attempt to solve $\tilde{P}(M;s^k)$ using the big-M simplex method. If the parametric problem is unbounded, conclude that $\tilde{P}(M)$ is inbounded for all $M \geq 0$. Otherwise obtain a solution z^{k+1}. If $F_M(z^k,s^k) = F_M(z^{k+1},s^k)$, stop: (z^k,s^k) is an equilibrium point.
3. Solve $\tilde{P}(M;z^{k+1})$ using simplex method to obtain a solution s^{k+1}. If $F_M(z^{k+1},s^k) = F_M(z^{k+1},s^{k+1})$, stop: (z^{k+1},s^k) is an equilibrium point. Otherwise set $k = k+1$ and return to step 2.

We note that problems $\tilde{P}(M;z^k)$ and $\tilde{P}(M;s^k)$ cannot be infeasible because $S \times \tilde{Z} \neq \emptyset$. Also, $\tilde{P}(M;z^k)$ is not unbounded since $(z^k)^T s \geq 0$ for all $s \in S$. However, it is possible to have $P(M;s^k)$ unbounded for all $M \geq 0$ meaning that the parametric problem is unbounded.

Proposition 10 *Algorithm 1 halts after a finite number of steps.*

Proof: Let be (z^k,s^k) and (z^{k+1},s^{k+1}) two consecutive points of the sequence generated by the algorithm. By step 2, z^{k+1} solves $\tilde{P}(M;s^k)$. Thus,

$$(s^k)^T z^{k+1} \leq (s^k)^T z^k. \tag{1.10}$$

If the algorithm does not stop at step 3, we have that $F_M(z^{k+1},s^{k+1}) \neq F_M(z^{k+1},s^k)$ for some $M \geq 0$. As s^{k+1} solves $\tilde{P}(M;z^{k+1})$, it must be $F_M(z^{k+1},s^{k+1}) > F_M(z^{k+1},s^k)$. So, it follows that

$$(s^{k+1})^T z^{k+1} < (s^k)^T z^{k+1}. \tag{1.11}$$

Therefore, by (1.11)-(1.10), we deduce that the points cannot be repeated. Since $\tilde{Z}_v \times S_v$ is a finite set, we conclude that the sequence $\{(z^k,s^k)\}$ is finite. ■

Proposition 11 *Algorithm 1 finds an equilibrium point of the penalized problem or it verifies that* $\tilde{P}(M)$ *is unbounded for all* $M \geq 0$.

Proof: By proposition 10, the algorithm halts. Suppose, at first, that it stops at step 2, because $\tilde{P}(M; s^k)$ is unbounded for all $M \geq 0$. So, $\tilde{P}(M)$ is unbounded for all $M \geq 0$. Now suppose that the algorithm stops at step 2 due to $F_M(z^k, s^k) = F_M(z^{k+1}, s^k)$. Note that s^k solves $\tilde{P}(M; z^k)$ and z^{k+1} solves $\tilde{P}(M; s^k)$. Then, $F_M(z^k, s^k) \geq F_M(z^k, s)$ for all $s \in S$ and $F_M(z^k, s^k) = F_M(z^{k+1}, s^k) \geq F_M(z, s^k)$ for all $z \in \tilde{Z}$. Therefore, (z^k, s^k) is an equilibrium point. Finally, the last stopping rule is $F_M(z^{k+1}, s^k) = F_M(z^{k+1}, s^{k+1})$ (step 3). Similarly to the previous case, we can conclude that (z^{k+1}, s^k) is an equilibrium. ■

A simpler version of algorithm 1 can be used to find an equilibrium point of the penalized problem associated to (P). In fact, by proposition 1, we deduce that $(s^k)^T z^{k+1} = 0$ for $k = 0$. Hence, algorithm 1 could stop at the initial iteration of step 2 returning (z^1, s^0) as an equilibrium point or concluding that $P(M)$ is unbounded for all $M \geq 0$.

3.2 GLOBAL ALGORITHM

In this subsection we propose an iterative algorithm for finding an ε-global solution to problem (LBP). At iteration $k = 0$, we consider problem (P). The procedure finds an equilibrium point (z^0, s^0) of the penalized problem $P(M)$, which is a local solution of (P). Then, a cut is added. At each iteration $k \geq 1$, we consider the problem $(\tilde{P})$ and the penalized problem $\tilde{P}(M)$, where $\tilde{Z} = Z \cap \{z \in \mathcal{R}^n : c^T z \geq c^T z^{k-1} + \varepsilon\}$. We note that this kind of cut is also used in [1, 11, 14, 15]. Then, a new equilibrium point (z^k, s^k) is obtained. If $(z^k)^T s^k = 0$ the cut is updated. Otherwise we apply an outer approximation procedure [10, section IX.1.5] to the following bilinear problem:

$$(PC) \quad \min s^T z \quad \text{s.t.} \quad (z, s) \in \tilde{Z} \times S.$$

Note that (PC) cannot be unbounded. Moreover, if (PC) is infeasible or its optimal value is nonzero, then $(\tilde{P})$ is infeasible. Otherwise, any optimal solution to (PC) is feasible to $(\tilde{P})$.

We can stop the outer approximation procedure as long as it gets a point $(\hat{z}, \hat{s})$ such that $\hat{z}^T \hat{s} < \bar{z}^T \bar{s}$. Then, we restart the equilibrium point algorithm from $\hat{z}$ or $\hat{s}$. Actually, at each iteration of the outer approximation method we determine an upper bound for (PC) given by a feasible point $(\hat{z}, \hat{s})$ (see [9, section 3.5.2]). If $\hat{z}^T \hat{s} < \bar{z}^T \bar{s}$, the procedure stops and returns $(\hat{z}, \hat{s})$. If this is never the case, the procedure finds out that $(\bar{z}^T \bar{s})$ is a solution to (PC) and returns this point.

Algorithm 2 (Global Algorithm)

1. If $S = \emptyset$, go to step 6. Otherwise, set $\tilde{Z} = Z$ and $k = 0$.
2. If $\tilde{Z} = \emptyset$, go to step 6. Otherwise, find $\hat{z} \in \tilde{Z}$.
3. Apply algorithm 1 from $\hat{z}$. If $\tilde{P}(M)$ is umbounded for all $M \geq 0$, stop: (LBP) is unbounded. Otherwise, let $(\bar{z}, \bar{s})$ be the returned equilibrium point.
4. If $\bar{s}^T \bar{z} = 0$, then $(\bar{z}, \bar{s})$ is a local solution to (P); set $(z^\star, s^\star) = (\bar{z}, \bar{s})$; apply a linear cut defining $\tilde{Z} = Z \cap \{z \in \mathcal{R}^n : c^T z \geq c^T z^\star + \varepsilon\}$, with $\varepsilon > 0$; set $k = k + 1$ and return to step 2.
5. If $\bar{s}^T \bar{z} > 0$, apply the modified *outer approximation* procedure to obtain a point $(\hat{z}, \hat{s})$. If $\hat{s}^T \hat{z} \geq \bar{s}^T \bar{z}$, go to step 6. Otherwise, go to step 3.
6. Stop. If $k = 0$, (LBP) is infeasible. Otherwise, $z^\star$ is an ε-global solution to (LBP).

Let us make some remarks about algorithm 2. At iteration $k = 0$, phase I of the simplex method may be used to check whether $\tilde{Z}$ or S are empty sets as well as to obtain the initial point $\hat{z} \in \tilde{Z}$. At iteration $k \geq 1$, it is better to apply the dual-simplex method at step 2. Finally, we note that it is more interesting to use a version of algorithm 1 that starts from a point $\hat{s} \in S$ imediately after returning from step 5 to step 3.

Proposition 12 *The global algorithm halts after a finite number of steps.*
Proof: Suppose, by contradiction, that an infinite loop occurs. Since each new cut discards at least one more vertex of Z, we deduce that step 4 is no longer executed after some iteration. Then, the loop runs between steps 3 and 5. So, an infinite sequence of vertices of $\tilde{Z} \times S$ is generated by step 3. Thus, some points are repeated.

Let $(\bar{z}, \bar{s})$ and $(\hat{z}, \hat{s})$ in $\tilde{Z} \times S$ be the equilibrium point generated by step 3 and the point returned by the outer approximation at a certain iteration of the infinite loop. At step 5, $\hat{s}^T \hat{z} < \bar{s}^T \bar{z}$ and algorithm 1 is applied from $z^0 = \hat{z}$. Then, the point s^0, obtained by step 1 of algorithm 1, satisfies $(s^0)^T \hat{z} \leq \hat{s}^T \hat{z}$. Let $(\bar{z}', \bar{s}')$ be the next equilibrium point generated. By (1.10)-(1.11) we conclude that $(\bar{s}')^T \bar{z}' \leq (s^0)^T \hat{z}$. By the last three inequalities it follows that the sequence cannot have repeated points, which is a contradiction. ∎

Proposition 13 *The global algorithm finds an ε-global solution to (LBP) or it verifies that the problem is umbounded or infeasible.*
Proof: By proposition 12, the algorithm stops. According to proposition 11, the stopping rule of step 3 guarantees that $\tilde{P}(M)$ is unbounded for all $M \geq 0$ and so is $P(M)$. Then, by theorem 5, (P) is unbounded. If the algorithm halts at step 6 with $k = 0$, we clearly have that (P) is infeasible. In these two cases, the same conclusions hold for (LPB) due to its equivalence with (P). Finally, the algorithm may stop when there is no more feasible points to (P) in the set $\tilde{Z} \times S$, where $\tilde{Z} = Z \cap \{z \in \mathcal{R}^n : c^T z \geq c^T z^\star + \varepsilon\}$. This case means that $z^\star$ is an ε-global solution to (LBP). ∎

Table 1.1 Computational results for random test problems

$n_1 + n_2$	n_2	m	$\tau = 0.1$	$\tau = 0.8$
50	12	15,17,19	0.13	0.04
50	38	15,17,19	4.36	1.87
60	15	18,20,22	0.26	0.02
60	45	18,20,22	31.80	10.51
70	17	20,22,24	0.40	< 0.01
70	53	20,22,24	37.70	10.49*
80	20	23,25,27	0.59	0.09
80	60	23,25,27	19.56	18.11
90	22	25,27,29	1.19	1.96
90	68	25,27,29	45.11	23.80*

3.3 COMPUTATIONAL RESULTS

The global algorithm was implemented in C and was run on a Pentium III, 600MHz. The value of ε was fixed at 0.013. Computational results for some random test problems are presented in the following.

The test problems were obtained with a method developed in [5]. The method allow us to control the density and the conditioning number of the constraint matrix and the number of local and global optima of the problem. Actually, it first generates a separable linear bilevel problem with the desired dimensions and number of optima. Then, it disguises the separability with a matrix transformation $HD^{-1}H$ while keeping the number of optima. The invertible diagonal matrix D and the block-diagonal Householder matrix H are randomly generated so that 10^{δ} is the conditioning number of D and τ is the percentual of nonzero entries in the basis vector of H, where δ and τ are given parameters (see [5] for details). Following suggestions of the authors, we introduced some modifications in the method to reduce the number of constraints. We also added nonnegativeness constraints for variables.

Table 1.1 presents computational results for 60 test problems. Each number in the last two columns is the average elapsed time in seconds for three problems with $\tau = 0.1$ or $\tau = 0.8$. The i-th problem ($i = 1, 2, 3$) is related to the i-th number in column m. It has 2^i local solutions (one of them is a global one) before considering $x \geq 0$ and $y \geq 0$. We used $\delta = 1$.

These results show that the algorithm had a good behavior in general. It was slightly affected by the density of the constraint matrix. On the other hand, we can observe that times increase very much when more variables are assigned to the second level. We remark that (*) indicates that one of the problems could not be solved in a limit time of 900s. This difficulty is due to the outer approximation procedure. For a future work it is interesting to consider alternatives to the outer approximation procedure.

Acknowledgments

We would like to thank CAPES and CNPq grant 302393/85-4(RN).

References

[1] M. Amouzegar and K. Moshirvazari. A penalty method for linear bilevel programming problems. Nonconvex Optimization and its Applications 20: 251–271, Kluwer, 1998.

[2] J. Bard. An efficient point algorithm for a linear two-stage optimization problem. *Operations Research*, 31:670–684, 1983.

[3] J. Bard. An investigation of the linear three level programming problem. *IEEE Trans. Systems, Man and Cybernetics*, 14:711–717, 1984.

[4] M. Bazaraa, J. Jarvis, and H. Sherali. *Linear programming and network flows*. Wiley, New York, 2nd edition, 1990.

[5] P. Calamai and L. Vicente. Generating linear and linear-quadratic bilevel programming problems. *SIAM J. Scientific and Stat. Computing*, 14:770–782, 1993.

[6] M. Campêlo, S. Dantas, and S. Scheimberg. A note on a penalty function approach for solving bilevel linear programs. *J. Global Optimization*, 16:245–255, 2000.

[7] M. Campêlo and S. Scheimberg. Characterizing local and global solutions of a linear bilevel problem. Technical Report ES-515/99, Univ. Federal do Rio de Janeiro, COPPE/Engenharia de Sistemas e Computação, 1999.

[8] P. Hansen, B. Jaumard, and G. Savard. New branch-and-bound rules for linear bilevel programming. *SIAM J. Scientific and Stat. Computing*, 13:1194–1217, 1992.

[9] R. Horst, P. Pardalos, and N. Thoai. *Introduction to global optimization*, volume 3 of *Nonconvex Optimization and Its Aplications*. Kluwer, 1995.

[10] R. Horst and H. Tuy. *Global optimization: deterministic approaches*. Spring-Verlag, Berlin, 1990.

[11] J. Júdice and A. Faustino. A sequential LCP method for bilevel linear programming. *Annals Operations Research*, 34:89–106, 1992.

[12] H. Konno. A cutting plane algorithm for solving bilinear programs. *Math. Programming*, 11:14–27, 1976.

[13] A. Migdalas, P. Pardalos, and P. Värbrand, editors. *Multilevel Optimization: algorithms and applications*, volume 20 of *Nonconvex Optimization and Its Aplications*. Kluwer, 1998.

[14] H. Önal. A modified simplex approach for solving bilevel linear programming problems. *Eur. J. Operational Research*, 67:126–135, 1993.

[15] H. Tuy, A. Migdalas, and P. Värbrand. A global optimization approach for the linear two-level program. *J. Global Optimization*, 3:1–23, 1993.

[16] L. Vicente and P. Calamai. Bilevel and multilevel programming: a bibliography review. *J. Global Optimization*, 5:291–306, 1994.

[17] D. White and G. Anandalingam. A penalty function approach for solving bi-level linear programs. *J. Global Optimization*, 3:397–419, 1993.