# A simplex approach for finding local solutions of a linear bilevel program by equilibrium points


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**Abstract**

In this paper, a linear bilevel programming problem (LBP) is considered. Local optimality conditions are derived. They are based on the notion of equilibrium point of an exact penalization for LBP. It is described how an equilibrium point can be obtained with the simplex method. It is shown that the information in the simplex tableaux can be used to get necessary and sufficient local optimality conditions for LBP. Based on these conditions, a simplex type algorithm is proposed, which attains a local solution of LBP by moving in equilibrium points. A numerical example illustrates how the algorithm works. Some computational results are reported.



**Keywords:** Bilevel linear programming, local optimization, simplex method.


## 1 Introduction

We consider the following linear bilevel program:

$$
\begin{array}{lll}
\text{(LBP)} & \max_{x,y} & f_1(x,y) = c_1^T x + c_2^T y \\
 & \text{s.t.} & x \geq 0, y \text{ solves:} \\
 & & \quad \begin{array}{ll} \max_y & f_2(x,y) = d^T y \\ \text{s.t.} & A_1 x + A_2 y \leq a, \\ & y \geq 0, \end{array}
\end{array}
$$

where $c_1, x \in \mathcal{R}^{n_1}$, $c_2, d, y \in \mathcal{R}^{n_2}$, $a \in \mathcal{R}^m$, $A_1 \in \mathcal{R}^{m \times n_1}$ and $A_2 \in \mathcal{R}^{m \times n_2}$.

This formulation has been extensively studied in the literature. Basic properties and solution methods can be found in [2, 6, 9, 12, 19], for instance. Also, we refer to [14] for an approach in the framework of mathematical programs with equilibrium constraints (MPEC). In terms of applications, bilevel programming has been used in many domains, e.g. network design, transportation, game theory, engineering and economics. Some examples are presented in [15, 18].

Problem LBP is a strongly NP-hard problem [11]. The main difficulties are due to the non-convexity of its feasible region, which is a connected set comprising faces of the polyhedron

$\{(x,y) \geq 0 : A_1x + A_2y \leq a\}$ [5]. This makes LBP a nonconvex optimization program possibly having many local optimal solutions.

In [10], we derive local optimality conditions for LBP based on the notion of equilibrium points of an exact penalization. The equilibrium requirement is shown to be a necessary optimality condition that is not difficult to attain. It suffices to solve two linear programs. On the other hand, it is generally harder to evaluate extra conditions to ensure that an equilibrium point gives a local optimal solution of LBP.

In this work, we derive other necessary and/or sufficient conditions, besides of those presented in [9, 10], based on the same notion of equilibrium. We use the information available in the optimal simplex tableaux related to the two linear programs solved. Usual nondegeneracy assumptions are considered to obtain conditions which are more attractive from a computational point of view.

The paper is organized as follows. In section 2, we present some basic tools to be used in our development. In subsection 2.1, we define the penalized problem related to LBP. Also, we present the notation and obtain preliminary local properties. In subsection 2.2, we recapitulate some results about equilibrium points. In section 3, we derive optimality conditions. In section 4, we propose an algorithm for finding a local optimal solution and present an illustrative example. Finally, we have a concluding remarks section where we comment about computational aspects of the local algorithm.

## 2 Basic Results

### 2.1 Preliminaries

Here, we consider an exact penalization for LBP that is used to derive optimality conditions. Also, we describe the adopted notation and obtain basic local results.

The penalized problem is obtained by replacing the inner linear problem of LBP by its KKT conditions and by penalizing the complementarity constraints into the objective function with a parameter $M \geq 0$. Thus, we have the following parametric bilinear program:

$$\begin{array}{lll} P(M) & \max & c_1^T x + c_2^T y - M(u^T w + v^T y) \\ & \text{s.t.} & A_1 x + A_2 y + w = a, \\ & & x \geq 0, y \geq 0, w \geq 0, \\ & & A_2^T u - v = d, \\ & & u \geq 0, v \geq 0, \end{array}$$

where $w \in \mathcal{R}^m$ is the primal slack vector and $u \in \mathcal{R}^m$ and $v \in \mathcal{R}^{n_2}$ are dual variable vectors. It is worth noting that other ways of penalizing the complementarity constraints have been considered in the literature, see [14] for instance.

Program $P(M)$ is an exact penalization for LBP. Actually, we show in [9] that LBP and $P(M)$ are both infeasible or unbounded for all $M \geq 0$ or else there is a correspondence between the global optimal solutions of the problems for all $M \geq M_0$, for some finite value $M_0 \geq 0$. This equivalence has supported some methods for finding global optimal solutions of LBP, see e.g. [2, 3, 9, 19].

More recently, this penalized problem has also been used to characterize local optimal solutions of LBP [10]. The optimality conditions are based on the concept of equilibrium point of $P(M)$, which was introduced in [8].

In this work, we further study the relations between equilibrium points of $P(M)$ and local optimal solutions of LBP. We derive optimality conditions which are more attractive from a computational point of view than those presented in [10].

We adopt the following notation. We consider the block matrices $A = [A_1\, A_2\, I_m] \in \mathcal{R}^{m\times n}$, $D = [0\ -I_{n_2}\ A_2^T] \in \mathcal{R}^{n_2\times n}$, $c^T = (c_1^T, c_2^T, 0) \in \mathcal{R}^n$, $z^T = (x^T, y^T, w^T) \in \mathcal{R}^n$ and $s^T = (0, v^T, u^T) \in \mathcal{R}^n$, where $n = n_1+n_2+m$, $I_p$ is the $(p\times p)$-identity matrix and 0 is a null matrix of appropriate dimension for each case. We define the sets $Z = \{z \in \mathcal{R}^n : Az = a, z \geq 0\}$ and $S = \{s \in \mathcal{R}^n : Ds = d, s \geq 0\}$. Thus, the penalized problem is rewritten as:

$$P(M) \quad \max\ F_M(z, s) = c^T z - M s^T z \quad \text{s.t.} \quad z \in Z, s \in S$$

We denote by $X_v$ the vertex set of a polyhedron $X$. Given a matrix $Q \in \mathcal{R}^{p\times q}$ and a subset $K \subset \{1, 2, \ldots, q\}$, we denote by $Q_K$ the submatrix that comprises the columns of $Q$ indexed by $K$. The columns of $Q_K$ are referred to by the original indices used in $Q$.

To develop the local analysis, we consider neighborhoods given by the infinity norm. Let us recall that the infinity norm of $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is $\| \nu \|_\infty = \max\{|\nu_i| : 1 \leq i \leq p\}$ and $B_\varepsilon(\bar\nu) = \{\nu \in \mathcal{R}^p : \| \nu - \bar\nu \|_\infty \leq \varepsilon\}$ is an $\varepsilon$-neighborhood of $\bar\nu \in \mathcal{R}^p$. Observe that $B_\varepsilon(\nu, \omega) = B_\varepsilon(\nu) \times B_\varepsilon(\omega)$ for any $(\nu, \omega) \in \mathcal{R}^p \times \mathcal{R}^q$.

Finally, we use the following point-to-set functions:

$$S(z) = \{s \in S : z^T s = 0\} \qquad \text{and} \qquad Z(s) = \{z \in Z : s^T z = 0\},$$

which map a point $z \in Z \subset \mathcal{R}^n$ (resp. $s \in S \subset \mathcal{R}^n$) into a polyhedron $S(z) \subset S$ (resp. $Z(s) \subset Z$). The vertex sets of polyhedra $S(z)$ and $Z(s)$ are $S_v(z) = S(z) \cap S_v$ and $Z_v(s) = Z(s) \cap Z_v$, respectively.

These functions were introduced in [8] and are closely related to our problem. Note that the feasible set of LBP is the domain of $S(\cdot)$, $\{z \in Z : S(z) \neq \emptyset\}$, or equivalently, the image of $Z(\cdot)$, $\{z \in Z : z \in Z(s) \text{ for some } s \in S\}$. Moreover, the next useful local property can be found in [10].

**Lemma 1** *The following assertions hold:* **(1)** *If $\bar{z} \in Z$ then there is $\varepsilon > 0$ such that $S(z) \subseteq S(\bar{z})$ and $S_v(z) \subseteq S_v(\bar{z})$ for all $z \in Z \cap B_\varepsilon(\bar{z})$.* **(2)** *If $\bar{s} \in S$ then there is $\varepsilon > 0$ such that $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for all $s \in S \cap B_\varepsilon(\bar{s})$.* **(3)** *If $(\bar{z}, \bar{s}) \in Z \times S$ then there is $\varepsilon > 0$ such that $S(z) \subseteq S(\bar{z})$, $S_v(z) \subseteq S_v(\bar{z})$, $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for all $(z, s) \in (Z \times S) \cap B_\varepsilon(\bar{z}, \bar{s})$.*

A direct consequence of the above lemma is the following characterization of polyhedra $S(\bar{z})$ and $Z(\bar{s})$ and their vertex sets, for each $\bar{z} \in Z$ and $\bar{s} \in S$.

**Corollary 2** *The following assertions hold:* **(1)** *If $\bar{z} \in Z$ ($\bar{z} \in Z_v$) then there is $\varepsilon > 0$ such that $S(\bar{z}) = \bigcup_{z \in Z \cap B_\varepsilon(\bar{z})} S(z)$ $\left(S_v(\bar{z}) = \bigcup_{z \in Z \cap B_\varepsilon(\bar{z})} S_v(z)\right)$.* **(2)** *If $\bar{s} \in S$ ($\bar{s} \in S_v$) then there is $\varepsilon > 0$ such that $Z(\bar{s}) = \bigcup_{s \in S \cap B_\varepsilon(\bar{s})} Z(s)$ $\left(Z_v(\bar{s}) = \bigcup_{s \in S \cap B_\varepsilon(\bar{s})} Z_v(s)\right)$.*

## 2.2 Equilibrium points

Now, we present the notion of equilibrium point of $P(M)$ and its relation to local optimal solutions of LBP.

**Definition 3** *A point $(\bar{z}, \bar{s})$ is an* **equilibrium point** *of the penalized problem $P(M)$ if there is $\bar{M} \geq 0$ such that, for each $M \geq \bar{M}$, it holds*

$$\max\{F_M(\bar{z}, s) : s \in S\} = F_M(\bar{z}, \bar{s}) = \max\{F_M(z, \bar{s}) : z \in Z\}. \tag{1}$$

An equilibrium point eliminates the penalty term thus giving a feasible solution of LBP, as stated in the next result [10].

**Lemma 4** *If $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem $P(M)$, then* $\min\{\bar{z}^T s : s \in S\} = \min\{\bar{s}^T z : z \in Z\} = \bar{s}^T \bar{z} = 0$.

Furthermore, the equilibrium equation (1) is a necessary condition for optimality in LBP. Actually, local optimal solutions of LBP are characterized in [10] as follows:

**Theorem 5** *A point $\bar{z}$ is a local optimal solution of LBP if, and only if, $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is an equilibrium point of the penalized problem $P(M)$ for all vertex $s \in S_v(\bar{z})$.*

In the next section, we are going to specialize the above theorem. We aim to find local optimality conditions which are more easily checked.

**Remark 6** *Let us consider the MPEC* $(P)$ *given by* $\max\,\{c^T z : z \in Z, s \in S, s^T z = 0\}$*. It holds that* $(P)$ *and* $P(M)$ *have the same global optimal solution set for all* $M \geq M_0$*, for some* $M_0 \geq 0$ *[9]. Additionally, an equilibrium point of* $P(M)$ *is a local optimal solution of* $(P)$ *and vice-versa [10]. Hence,* $(P)$ *and LBP have the same global optimal solution set but the local optimal solution set of LBP may be strictly included in the local optimal solution set of* $(P)$*.*

To conclude this subsection, we present an algorithm to find an equilibrium point by solving two linear programs [8]. Regarding Definition 3, let us define the following parametric linear problems:

$$\begin{array}{lll} P(M,\bar{s}) & \max & F_M(z,\bar{s}) = c^T z - M\bar{s}^T z \\ & \text{s.t.} & z \in Z = \{z \geq 0 : Az = a\} \\[1ex] P(M,\bar{z}) & \max & F_M(\bar{z},s) = c^T \bar{z} - M\bar{z}^T s \\ & \text{s.t.} & s \in S = \{s \geq 0 : Ds = d\} \end{array}$$

Thus, $(\bar{z},\bar{s})$ is an equilibrium point if, and only if, $\bar{z}$ and $\bar{s}$, respectively, are solutions of $P(M,\bar{s})$ and $P(M,\bar{z})$ for all $M \geq \bar{M}$.

Let us note that these problems can be solved by the simplex method. In fact, problem $P(M,\bar{s})$ can be solved by implicitly considering parameter $M$. Note that its linear parametric objective function $F_M(z,\bar{s}) = c^T z - M\bar{s}^T z$ can be maximized as in the *big-M* simplex method, assuming that $M$ is a dominating value [4]. On the other hand, the solution point of problem $P(M,\bar{z})$ does not depend on parameter $M$, since the first term of its objective funtion $F_M(\bar{z},s) = c^T\bar{z} - M\bar{z}^T s$ is a constant.

The following algorithm, given in [8], finds an equilibrium point of $P(M)$.

**Algorithm 1** Basic Equilibrium Point Algorithm

0. If $Z \times S = \emptyset$ then LBP is infeasible. Otherwise, let $z^0 \in Z$.
1. Solve problem $P(M,z^0)$, to obtain a solution $\bar{s}$;
2. Attempt to solve problem $P(M,\bar{s})$. If this parametric problem is unbounded, then $P(M)$ is unbounded for all $M \geq 0$ and so is LBP. Otherwise, obtain a solution $\bar{z}$. The point $(\bar{z},\bar{s})$ is an equilibrium point.

This algorithm is well-defined, according to Proposition 1 given in [9] and Lemma 4. Additionally, a similar algorithm that starts with a point $s^0 \in S$ can be designed. Details can be found in

[8]. These procedures have been successfully used to improve performance of branch-and-bound algorithms for LBP [16].

Let us note the Basic Equilibrium Point Algorithm corresponds to one iteration of the Mountain Climbing Algorithm, that was proposed by Konno [13] to find a KKT point of a bilinear problem. Such iterative procedure has also been used by Audet *at al.* [1] to calculate lower bounds in a branch-and-bound method for bilinear programming.

## 3 Local optimality conditions

Local optimal solutions of LBP are associated with equilibrium points of $P(M)$ according to Theorem 5. In this section, we study other local optimality conditions for LBP in terms of equilibrium points. We derive properties which are computationally simpler to check.

From now on, we treat $(\bar{z}, \bar{s})$ as an equilibrium point of $P(M)$. Thus, $\bar{z}$ and $\bar{s}$, respectively, are solutions of the parametric linear problems $P(M, \bar{s})$ and $P(M, \bar{z})$. As solutions of these problems are attained at vertices, we will assume that $(\bar{z}, \bar{s}) \in Z_v \times S_v$. We denote by $J$ the set of indices $\{1, 2, \ldots, n\}$.

Let us consider below the initial and the optimal simplex tableau of $P(M, \bar{s})$, where $B \subset J$ and $N \subset J$, respectively, are the index sets of basic and nonbasic variables at $\bar{z}$.

$$
\begin{array}{c|cc|c}
 & z_B^T & z_N^T & \\ \hline
 & A_B & A_N & a \\ \hline
M & -\bar{s}_B^T & -\bar{s}_N^T & 0 \\ \hline
 & c_B^T & c_N^T & 0
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|cc|c}
 & z_B^T & z_N^T & \\ \hline
 & I & \tilde{A}_N = A_B^{-1}A_N & \bar{z}_B = A_B^{-1}a \\ \hline
M & 0 & -\tilde{s}_N^T = -\bar{s}_N^T + \bar{s}_B^T\tilde{A}_N & 0 = \bar{s}_B^T\bar{z}_B \\ \hline
 & 0 & \tilde{c}_N^T = c_N^T - c_B^T\tilde{A}_N & -c_B^T\bar{z}_B
\end{array}
$$

In the above tableaux, the objective function $F_M(z, \bar{s}) = c^T z - M\bar{s}^T z$ is represented in two rows. The first one corresponds to the complementarity term $\bar{s}^T z$. The second one expresses the linear term $c^T z$. As observed before, it is possible to implicitly consider the parameter $M$ by optimizing with priority the first row, as does the *big-M* simplex method.

Now, let us consider the initial and the optimal simplex tableaux of $P(M, \bar{z})$, where $E \subset J$ and $R \subset J$, respectively, are the index sets of basic and nonbasic variables at $\bar{s}$.

$$
\begin{array}{c|cc|c}
 & s_E^T & s_R^T & \\
\hline
 & D_E & D_R & d \\
\hline
M & -\bar{z}_E^T & -\bar{z}_R^T & 0
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|cc|c}
 & s_E^T & s_R^T & \\
\hline
 & I & \tilde{D}_R = D_E^{-1} D_R & \bar{s}_E = D_E^{-1} d \\
\hline
M & 0 & -\tilde{z}_R^T = -\bar{z}_R^T + \bar{z}_E^T \tilde{D}_R & 0 = \bar{z}_E^T \bar{s}_E
\end{array}
$$

In problem $P(M;\bar{z})$, the function to be maximized is $F_M(\bar{z}, s) = c^T\bar{z} - M\bar{z}^T s$. As the first term is a constant, we represent only the second term in the tableaux. In this case, the solution does not depend on parameter $M$.

Since $(\bar{z}, \bar{s})$ is an equilibrium point and $A_B$ and $D_E$ are optimal bases, we have the results below:

**Property 7** *The following assertions hold:* **(1)** $\tilde{s}_N \geq 0$ *and* $\tilde{z}_R \geq 0$. **(2)** *Given* $i \in N$, *if* $\tilde{c}_i > 0$ *then* $\tilde{s}_i > 0$.

*Proof:* (1) follows by Lemma 4. (2) holds since $\bar{z}$ is a solution of $P(M, \bar{s})$. This means that $\tilde{c}_i - M\tilde{s}_i \leq 0$ for all $i \in N$ and all $M \geq \bar{M} \geq 0$, implying that $\tilde{s}_i > 0$ if $\tilde{c}_i > 0$. ∎

Let us define the following sets:

$$N^+ = \{i \in N : \tilde{c}_i > 0\} \qquad \text{and} \qquad R^0 = \{j \in R : \tilde{z}_j = 0\},$$

that, respectively, refer to the improving directions of $Z$ at $\bar{z}$ related to the objective function $c^T z$ and to the directions of $S$ at $\bar{s}$ which are feasible to the constraint $\bar{z}^T s = 0$.

Sufficient optimality conditions for LBP are readily derived as follows:

**Property 8** *The following assertions hold:* **(1)** *If* $N^+ = \emptyset$ *then* $\bar{z}$ *is a global optimal solution of LBP.* **(2)** *If* $R^0 = \emptyset$ *then* $\bar{z}$ *is a local optimal solution of LBP.*

*Proof:* (1) holds because $\bar{z}$ is a solution for the leader's relaxation $\max\{c^T z : z \in Z\}$, if $N^+ = \emptyset$. (2) follows by Theorem 5 since $\bar{s}$ is the unique point in $S$ which is complementary to $\bar{z}$, i.e. $S(\bar{z}) = \{\bar{s}\}$, when $R^0 = \emptyset$. ∎

To study other optimality conditions, we consider below the extreme directions of $Z$ and $S$ at $\bar{z}$ and $\bar{s}$, respectively. These directions can be obtained in the optimal tableaux and are given by the columns of the following matrices:

$$G_N = \begin{bmatrix} -\tilde{A}_N \\ I_{n-m} \end{bmatrix} \in \mathcal{R}^{n \times (n-m)} \qquad \text{and} \qquad H_R = \begin{bmatrix} -\tilde{D}_R \\ I_{n-n_2} \end{bmatrix} \in \mathcal{R}^{n \times (n-n_2)},$$

where $I_p$ is a $p \times p$ identity matrix. For $i \in N$ and $j \in R$, column $G_i$ of $G$ and column $H_j$ of $H$ are:

$$G_i = \begin{bmatrix} -\tilde{A}_i \\ e_i \end{bmatrix} \in \mathcal{R}^n \qquad \text{and} \qquad H_j = \begin{bmatrix} -\tilde{D}_j \\ e_j \end{bmatrix} \in \mathcal{R}^n.$$

Hence, the components of these directions are given by

$$G_{ki} = \begin{cases} -\tilde{A}_{ki}, & \text{if } k \in B, \\ 0, & \text{if } k \in N \setminus \{i\}, \\ 1, & \text{if } k = i, \end{cases} \qquad \text{and} \qquad H_{kj} = \begin{cases} -\tilde{D}_{kj}, & \text{if } k \in E, \\ 0, & \text{if } k \in R \setminus \{j\}, \\ 1, & \text{if } k = j. \end{cases} \tag{2}$$

Note that the components of the reduced cost vectors are:

$$\tilde{c}_i = c^T G_i, \quad \tilde{s}_i = \bar{s}^T G_i \geq 0 \quad \text{and} \quad \tilde{z}_j = \bar{z}^T H_j \geq 0, \tag{3}$$

for $i \in N$ and $j \in R$, where the inequalities hold by Property 7(1). So, sets $N^+$ and $R^0$ can be redefined accordingly. It will also be useful to define the subsets:

$$N^{+0} = \{i \in N^+ : s^T G_i = 0 \text{ for some } s \in S(\bar{z})\} \quad \text{and} \quad R^{00} = \{j \in R^0 : \bar{z}_j = 0\}.$$

Next we show that the points of $Z$ can be expressed in terms of $\bar{z}$ and the directions $G_i$. A similar development is then applied to $S$, $\bar{s}$ and the directions $H_j$.

**Lemma 9** *If $z = (z_1, z_2, \ldots, z_n) \in Z$ then $z = \bar{z} + \sum_{i \in N} z_i G_i$.*

*Proof:* Let $z = (z_B^T, z_N^T)^T \in Z$. Then, $A_B z_B + A_N z_N = a$, and so $z_B = \bar{z}_B - \tilde{A}_N z_N$. Since $\bar{z}_N = 0$, it follows that $\begin{bmatrix} z_B \\ z_N \end{bmatrix} = \begin{bmatrix} \bar{z}_B \\ \bar{z}_N \end{bmatrix} + \begin{bmatrix} -\tilde{A}_N \\ I_{n-m} \end{bmatrix} z_N$. Therefore, $z = \bar{z} + \sum_{i \in N} z_i G_i$. ■

**Lemma 10** *If $s = (s_1, s_2, \ldots, s_n) \in S$, then $s = \bar{s} + \sum_{j \in R} s_j H_j$. Moreover, if $s \in S(\bar{z})$ then $s = \bar{s} + \sum_{j \in R^{00}} s_j H_j$.*

*Proof:* Like in Lemma 9, we find that $s = \bar{s} + \sum_{j \in R} s_j H_j$. Now assume that $s \in S(\bar{z})$, that is, $\bar{z}^T s = \bar{z}^T \bar{s} = 0$. Then, $s_j = 0$ if $\bar{z}_j > 0$. In addition, we have that $\sum_{j \in R} s_j \bar{z}^T H_j = 0$. Thus, the last inequality in (3) implies that $s_j = 0$ whenever $\tilde{z}_j = \bar{z}^T H_j > 0$. Therefore, the expression giving $s$ can be reduced to $s = \bar{s} + \sum_{j \in R^{00}} s_j H_j$. ■

In order to obtain optimality conditions for LBP, from now on we consider the following assumption:

[$\mathcal{D}$] If $s \in S(\bar{z})$, then $s^T G_N \geq 0$, i.e. $s^T G_i \geq 0$ for all $i \in N$.

This assumption trivially holds when $\bar{z}$ is a nondegenerate vertex. In this case $A_B$ is the unique basis given $\bar{z}$. Then, since $\bar{z}$ is a solution of $\min\{s^T z : z \in Z\}$, the optimal reduced cost $s^T G_N = s_N^T - s_B^T A_B^{-1} A_N$ is nonnegative.

Now we derive a sufficient condition for local optimality of the equilibrium point.

**Theorem 11** *If $N^{+0} = \emptyset$ then $\bar{z}$ is a local optimal solution of LBP.*

*Proof:* Suppose, by contradiction, that $\bar{z}$ is not a local optimal solution of LBP. Let $\varepsilon$ be given by Lemma 1. Then, there is $z \in Z \cap B_\varepsilon(\bar{z})$, with $\emptyset \neq S(z) \subseteq S(\bar{z})$ and $c^T z > c^T \bar{z}$. By Lemma 9, $z = \bar{z} + \sum_{i \in N} z_i G_i$. Thus, $0 < c^T(z - \bar{z}) = \sum_{i \in N} z_i c^T G_i$. Hence, there exists $k \in N$ with $z_k > 0$ and $c^T G_k > 0$. Therefore, $k \in N^+$. On the other hand, let $\hat{s} \in S(z) \subseteq S(\bar{z})$. Then, $0 = \hat{s}^T(z - \bar{z}) = \sum_{i \in N} z_i \hat{s}^T G_i$. By assumption $[\mathcal{D}]$, we conclude that $z_i \hat{s}^T G_i = 0$ for all $i \in N$. In particular, $\hat{s}^T G_k = 0$. Therefore, we have that $k \in N^{+0}$. ■

Actually, the sufficient condition presented in Theorem 11 is also necessary under an assumption of nondegeneracy, as shown in the next theorem. First, we introduce the following definition.

**Definition 12** *For $i \in N$, the direction $G_i$ is degenerate if $\inf_{k \in B} \left\{ \bar{z}_k / \tilde{A}_{ki} : \tilde{A}_{ki} > 0 \right\} = 0$. Additionally, the set $N^{+0}$ is said to be* **totally degenerate** *if $N^{+0} \neq \emptyset$ and $G_i$ is degenerate for all $i \in N^{+0}$.*

**Theorem 13** *If $\bar{z}$ is a local optimal solution of LBP then $N^{+0} = \emptyset$ or $N^{+0}$ is totally degenerate.*

*Proof:* Suppose, by contradiction, that $N^{+0}$ is nonempty and not totally degenerate. Then, there are $i \in N^{+0}$ and $s \in S(\bar{z})$ such that $G_i^T s = 0$ and $\bar{z} + \alpha G_i \in Z$ for all $0 < \alpha \leq \inf_{k \in B} \left\{ \bar{z}_k / \tilde{A}_{ki} : \tilde{A}_{ki} > 0 \right\}$. Let $\varepsilon > 0$ arbitrary. Thus, there is $z = \bar{z} + z_i G_i \in Z \cap B_\varepsilon(\bar{z})$ with $z_i > 0$. Since $z^T s = \bar{z}^T s + z_i G_i^T s = 0$, we have that $z \in B_\varepsilon(\bar{z})$ is feasible to LBP. Moreover, $c^T(z - \bar{z}) = z_i c^T G_i > 0$ because $i \in N^+$. Hence, $\bar{z}$ is not a local optimal solution of LBP. ■

According to Theorem 5, if $\bar{z}$ is not a local optimal solution of LBP, then $(\bar{z}, s)$ is not an equilibrium point for some $s \in S(\bar{z})$, $s \neq \bar{s}$. Corollary 14 gives more information about such points $s$ as follows.

**Corollary 14** *Assume that set $N^{+0}$ is not totally degenerate. If $\bar{z}$ is not a local optimal solution of LBP then, for all $i \in N^{+0}$ such that $G_i$ is not degenerate and all $\hat{s} \in \arg\min\{G_i^T s : s \in S(\bar{z})\}$, it holds that $(\bar{z}, \hat{s})$ is not an equilibrium point.*

*Proof:* Assume that $\bar{z}$ is not a local optimal solution of LBP. Let $i \in N^{+0}$ such that $G_i$ is not degenerate. There is always such an $i$, due to Theorem 11 and the fact that $N^{+0}$ is not totally degenerate. Then, there is $z_i > 0$ such that $z = \bar{z} + z_i G_i \in Z$. Let $\hat{s} \in \arg\min\{G_i^T s : s \in S(\bar{z})\} \neq \emptyset$. As $i \in N^{+0}$, it holds that $G_i^T \hat{s} = 0$. Thus, $(c - M\hat{s})^T(z - \bar{z}) = z_i c^T G_i > 0$ for all $M$. Hence, $(\bar{z}, \hat{s})$ is not an equilibrium point. ■

Corollary 14 assures that if the equilibrium point $(\bar{z}, \bar{s})$ does not give a local optimal solution of LBP, an alternative solution $\hat{s} \neq \bar{s}$ of $P(M, \bar{z})$ must result in an improved equilibrium point.

Let us note that we could redefine $N^{+0} = \left\{ i \in N^+ : \min\{G_i^T s : s \in S(\bar{z})\} = 0 \right\}$. Thus, to check the condition established in Theorem 11 it may be necessary to solve many linear programs. In some cases, this computational effort can be reduced according to the following result.

**Lemma 15** *Let $i \in N^+$. If $R^{00} = \emptyset$ or $G_i^T H_j \geq 0$ for all $j \in R^{00}$ then $i \notin N^{+0}$.*

*Proof:* Let $i \in N^+$. Suppose, by contradiction, that $i \in N^{+0}$. Then, there is $s \in S(\bar{z})$ such that $G_i^T s = 0$. Moreover, Property 7(2) implies that $G_i^T \bar{s} = \tilde{s}_i > 0 = G_i^T s$. By Lemma 10, it follows that $\sum_{j \in R^{00}} s_j G_i^T H_j < 0$. Therefore, $G_i^T H_j < 0$ for some $j \in R^{00}$. ∎

The above lemma together with Theorem 11 yield the following result:

**Corollary 16** *If the set $\{(i,j) \in N^+ \times R^{00} : G_i^T H_j < 0\}$ is empty, then $\bar{z}$ is a local optimal solution of LBP.*

**Remark 17** *It is not hard to compute the scalar product $G_i^T H_j$, for $i \in N^+$ and $j \in R^{00}$. Indeed, by expression (2), we have that $G_i^T H_j = \sum_{k \in B \cap E} \tilde{A}_{ki} \tilde{D}_{kj} + \Delta_{ij}$, where $\Delta_{ij} = 1$ (if $i = j$) or $\Delta_{ij} = H_{ij} + G_{ji}$ (if $i \neq j$). Moreover, few elements are expected in $B \cap E$, since $\bar{z}^T \bar{s} = 0$.*

We can still obtain optimality conditions for LBP which are easier to verify if we consider conventional nondegenerate assumptions. We study these cases below.

**Lemma 18** *The following assertions hold:* **(1)** *If $\bar{z}$ is nondegenerate then $s^T G_N = s_N$, for all $s \in S(\bar{z})$, $N^+ \subset N \cap E$, $R^{00} \subset N \cap R^0$ and $N^+ \cap R^{00} = \emptyset$.* **(2)** *If $\bar{s}$ is nondegenerate then $z^T H_R = z_R$, for all $z \in Z(\bar{s})$, and $R^{00} = R^0$.* **(3)** *If $\bar{z}$ and $\bar{s}$ are nondegenerate then $B \cap E = \emptyset$.*

*Proof:*

(1) Assume that $\bar{z}$ is a nondegenerate vertex, i.e. $\bar{z}_B > 0$. Let $s \in S(\bar{z})$. Since $\bar{z}^T s = 0$, it must be $s_B = 0$. It follows that $s^T G_N = s_N - s_B^T \tilde{A}_N = s_N$. In particular, $\tilde{s}_N = \bar{s}^T G_N = \bar{s}_N$. Thus, by Property 7(2), $\bar{s}_i = \tilde{s}_i > 0$ if $i \in N^+$. This yields that $N^+ \subset E$ and so $N^+ \subset N \cap E$. The inclusion $R^{00} \subset N \cap R^0$ comes from the definition of $R^{00}$ and the fact that $\bar{z}_B > 0$. In addition, we have that $N^+ \cap R^{00} \subset E \cap R^0 = \emptyset$.

(2) Given that $\bar{s}$ is nondegenerate, an argumentation similar to the first part of (1) shows that $z^T H_R = z_R$, for all $z \in Z(\bar{s})$. In particular, $\tilde{z}_R = \bar{z}^T H_R = \bar{z}_R$ implying that $R^{00} = R^0$.

(3) Since $\bar{z}_B > 0$, $\bar{s}_E > 0$ and $\bar{z}^T\bar{s} = 0$, it follows that $B \cap E = \emptyset$. ■

By Lemma 18(1), we have that $N^{+0} = \{i \in N^+ : \min\{s_i : s \in S(\bar{z})\} = 0\}$ when $\bar{z}$ is nondegenerate. Therefore, Theorems 11 and 13 result in the following characterization given in [10].

**Corollary 19** *Assume that $\bar{z}$ is nondegenerate. Then $\bar{z}$ is a local optimal solution of LBP if, and only if, $N^{+0} = \emptyset$, that is, the set $\{i \in N^+ : \min\{s_i : s \in S(\bar{z})\} = 0\}$ is empty.*

**Remark 20** *If $\bar{z}$ is nondegenerate and $(i, j) \in N^+ \times R^{00}$, then $G_i^T H_j = \sum_{k \in B \cap E'} \tilde{A}_{ki}\tilde{D}_{kj} - \tilde{D}_{ij}$, where $E' = \{k \in E : \bar{s}_k = 0\}$ is the index set of the basic degenerate variables at $\bar{s}$. Indeed, let us consider the expression given at Remark 17. First, since $\bar{z}^T\bar{s} = 0$ and $\bar{z}_B > 0$, it must be $E \backslash E' \subset N$ and so $B \cap E = B \cap E'$. In addition, by Lemma 18(1), $N^+ \cap R^{00} = \emptyset$ and $(i, j) \in E \times N$, which respectively imply $i \neq j$ and $H_{ij} + G_{ji} = -\tilde{D}_{ij}$.*

When both $\bar{z}$ and $\bar{s}$ are nondegenerate, necessary conditions for local optimality are much simpler. In fact, we can state simpler versions of Lemma 15 and Corollary 16 as follows.

**Lemma 21** *Assume that $\bar{z}$ and $\bar{s}$ are nondegenerate and let $i \in N^+$. If $R^0 = \emptyset$ or $\tilde{D}_{ij} \leq 0$ for all $j \in R^0$ then $i \notin N^{+0}$.*

*Proof:* By Remark 20 and Lemma 18, $G_i^T H_j = -\tilde{D}_{ij}$ for any $i \in N^+$ and $j \in R^{00} = R^0$. The result then follows by Lemma 15. ■

**Corollary 22** *Assume that $\bar{z}$ and $\bar{s}$ are nondegenerate. If the set $\{(i, j) \in N^+ \times R^0 : \tilde{D}_{ij} > 0\}$ is empty, then $\bar{z}$ is a local optimal solution of LBP.*

# 4 A local algorithm

In this section, we propose an algorithm for finding a local optimal solution of LBP. It uses the results obtained in the previous sections.

**Algorithm 2**

0. If $Z \times S = \emptyset$ then LBP is infeasible. Otherwise, let $z^0 \in Z$. Set $k = 1$.
1. Solve $P(M, z^0)$ by the simplex method to obtain a solution $\bar{s}^k \in S_v$.
2. Attempt to solve $P(M, \bar{s}^k)$ by the simplex method. If it is unbounded, then LBP is unbounded. Otherwise, let $\bar{z}^k \in Z_v$ be a solution. Then, $(\bar{z}^k, \bar{s}^k)$ is an equilibrium point.

3. Rergarding Lemmas 15 and 21, determine $N^{+0}$.

   3.1. If $N^{+0} = \emptyset$ then stop returning $\bar{z}^k$ as a local optimal solution of LBP.

   3.2. If $N^{+0} \neq \emptyset$ and is totally degenerate then stop with a failure, that is, there is no guarantee that the equilibrium point gives a local optimal solution of LBP.

   3.3. If $N^{+0} \neq \emptyset$ and is not totally degenerate, let $i \in N^{+0}$ such that $G_i$ is not degenerate.

4. Take $\bar{s}^{k+1} \in \arg\min\{G_i^T s : s \in S(\bar{z}^k)\}$, which is another solution of $P(M, \bar{z}^k)$. Set $k = k+1$ and go to step 2.

Let us note that we can apply the first phase of the simplex method to verify whether or not $Z$ and $S$ are both nonempty. If so, the initial point $z^0$ can be taken as the point in $Z$ achieved by the method. However, a good choice for $z^0$ may be given by the solution of the leader's relaxation $\max\{c^T z : z \in Z\}$, if it exists. Many algorithms proposed in the literature start from this point [11, 12, 17].

The proof of the correctness of Algorithm 2 essentially follows the results stated in section 3. Let us note that Algorithm 2 generates a sequence $\{(\bar{z}^k, \bar{s}^k)\}$ of equilibrium points such that $c^T z^{k+1} > c^T z^k$. The initial one ($k = 1$) is clearly obtained by Steps 1 and 2. The iterated ones ($k > 1$) are determined by Steps 4 and 2, which also comprise an equilibrium point procedure starting with $\bar{z}^k$. In fact, $\bar{s}^{k+1}$ is a solution of $P(M, \bar{z}^k)$ because $(\bar{z}^k)^T \bar{s}^{k+1} = 0$. Moreover, the next equilibrium point improves the objective function, provided that $i \in N^+$.

Two final remarks about Step 3 are worthwhile. First, once we have found the optimal tableau of $P(M, \bar{z}^k)$, we can use it to verify if $i \in N^{+0}$, that is, if there exists $s \in S(\bar{z}^k)$ such that $G_i^T s = 0$. Instead of solving the problem $\min\{G_i^T s : s \in S(\bar{z}^k)\}$, we can solve the parametric problem $\min\{G_i^T s + M(\bar{z}^k)^T s : s \in S\}$ from the optimal tableau of $P(M, \bar{z}^k)$. Indeed, we can apply the *big-M* simplex method from this tableau with the row related to the objective function properly including the coefficients given by $G_i$. Second, when $N^{+0}$ is totally degenerate, we consider that the algorithm fails because there is no guarantee whether $\bar{z}^k$ is a local optimal solution of LBP or not. Even so, the equilibrium condition ensures that $(\bar{z}^k, \bar{s}^k)$ is a local optimum of the MPEC $(P)$, according to Remark 6.

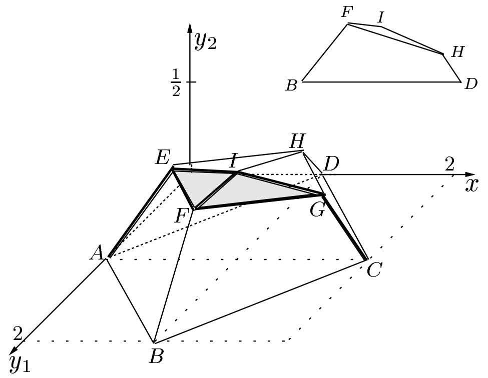


Figure 1: Illustration of the numerical example

### 4.1 Numerical Example

In order to demonstrate the algorithm, we consider the following example:

$$
\begin{array}{l}
\max\, f_1(x,y) = -x + 2y_1 - 20y_2 \\
\text{s.t. } \; x \geq 0,\; y = (y_1, y_2) \text{ solves} \\
\quad \max\, f_2(x,y) = -y_1 + 10y_2 \\
\quad \text{s.t.} \quad \begin{array}{rl}
x + y_1 + y_2 \leq 3, & 16x - \phantom{1}6y_1 + 60y_2 \leq 37 \\
x + y_1 - y_2 \geq 1, & 6x - 16y_1 + 60y_2 \leq 17 \\
-x + y_1 + y_2 \leq 1, & 6x - \phantom{1}6y_1 + 60y_2 \leq 27 \\
x - y_1 + y_2 \leq 1, & 16x - 16y_1 + 60y_2 \leq 27 \\
y_1 \geq 0,\; y_2 \geq 0 &
\end{array}
\end{array}
$$

In Figure 1, its feasible set is described by edges $AE$ and $GC$ and the triangules $EFI$ and $FGI$. We have that point $A$ is the global optimal solution and point $C$ is a local optimal solution. Also, note that point $B$ is the optimum of the leader's relaxation and that the points in edge $FI$ have the same value of the second level function.

After introducing the slack variables, we obtain the sets:

$$
\begin{aligned}
Z &= \{z \geq 0:\; z_1 + z_2 + z_3 + z_4 = 3,\; z_1 + z_2 - z_3 - z_5 = 1,\; -z_1 + z_2 + z_3 + z_6 = 1, \\
&\quad z_1 - z_2 + z_3 + z_7 = 1,\; 16z_1 - 6z_2 + 60z_3 + z_8 = 37,\; 6z_1 - 16z_2 + 60z_3 + z_9 = 17, \\
&\quad 6z_1 - 6z_2 + 60z_3 + z_{10} = 27,\; 16z_1 - 16z_2 + 60z_3 + z_{11} = 27\}, \\
S &= \{s \geq 0:\; s_2 - s_4 + s_5 - s_6 + s_7 + 6s_8 + 16s_9 + 6s_{10} + 16s_{11} = 1, \\
&\quad -s_3 + s_4 + s_5 + s_6 + s_7 + 60s_8 + 60s_9 + 60s_{10} + 60s_{11} = 10\}.
\end{aligned}
$$

Let us start the algorithm at $z^0 = (1,2,0,0,2,0,2,33,43,33,43)^T$, which corresponds to point $B$ in Figure 1. A basic solution $\bar{s}^1$ of $P(M, z^0)$ is such that $\bar{s}^1_E = (0, 1/6)^T$, for $E = \{9, 10\}$. Then, the solution set of $P(M, \bar{s}^1)$ is the edge $FI$. Let us consider $\bar{z}^1$ as the degenerate vertex $I$, given by $B = \{1,2,3,4,5,6,7,9\}$. Thus, $\bar{z}^1_B = (1,1,0.45,0.55,0.55,0.55,0.55,0)^T$. We have that $(\bar{z}^1,\bar{s}^1)$ is an equilibrium point, with $f_1(\bar{z}^1,\bar{s}^1) = -8$. By searching the optimal tableau, we get that $N^+ = \{10\}$ and $G^T_{10} = \frac{1}{150}\,(15,0,-4,-11,19,19,-11,0,150,150,0)$. Moreover, $G^T_{10}\bar{s}^2 = 0$, for a basic solution $\bar{s}^2$ of $P(M,\bar{z}^1)$, where $E = \{8,9\}$ and $\bar{s}^2_E = (1/6,0)^T$. This means that $N^{+0} = \{10\}$.

Returning to Step 2, the solution $\bar{z}^2$ of $P(M,\bar{s}^2)$ is attained at the degenerate point $G$. We have that $\bar{z}^2_B = (1.75,1,0.25,1.5,1.5,7.5,7.5,0)^T$, for $B = \{1,2,3,5,6,9,10,11\}$. Thus, we achieve a new equilibrium point $(\bar{z}^2,\bar{s}^2)$, with $f_1(\bar{z}^2,\bar{s}^2) = -4.75$. Again, we have one improving direction, which is indexed by $N^+ = \{8\}$. The tableau provides us with $G^T_8 = \frac{1}{44}(1,0,-1,0,2,2,0,44,54,54,44)$. Since a vertex $\bar{s}^3 \in S(\bar{z}^2)$ such that $G^T_8\bar{s}^3 = 0$ can be found, it holds that $N^{+0} = \{8\}$. Actually, $\bar{s}^3$ is a nondegenerate basic solution of $P(M,\bar{z}^2)$ that is given by $E = \{4,7\}$ and $\bar{s}^3_E = (4.5,5.5)^T$.

Once more we return to Step 2 to solve $P(M,\bar{s}^3)$. A nondegenerate basic solution $\bar{z}^3$ is attained at point $C$. Thus, we have that $\bar{z}^3_B = (2,1,2,2,11,21,21,11)^T$, for $B = \{1,2,5,6,8,9,10,11\}$. The third equilibrium point $(\bar{z}^3,\bar{s}^3)$ is then found, with $f_1(\bar{z}^3,\bar{s}^3) = 0$. It follows that $N^+ = \{7\}$ and $G^T_7 = \frac{1}{2}(-1,1,0,0,0,-2,2,22,22,12,32)$. This time, we can use Lemma 21 to ensure that $N^{+0} = \emptyset$. In fact, $R^0 = \{3\}$ and the corresponding entry in the tableau is $\tilde{D}_{73} = -0.5$. Hence, $\bar{z}^3$ is a local optimal solution.

Some final observations can be made in connection with the choices of the points or basis along the execution of the algorithm. First, note that $\bar{s}^2$ is another solution of $P(M,z^0)$. Thus, if we had chosen this solution at the first iteration, we would have achieved point $G$ right away. Second, another basic solution for $P(M,\bar{s}^1)$ would be vertex $F$. With this alternative option, the sequence of points generated would be the same as before (except for $F$ instead of $I$). Finally, different basis could have been chosen for representing point $I$ at the first iteration. Some of them may not be convenient. For instance, if we had obtained the tableau for $B = \{1,2,3,4,5,6,7,11\}$, we would have got the case where $N^{+0} = \{10\}$ would be totally degenerate.

# 5 Concluding remarks

We have developed a simplex framework for determining a local optimal solution of LBP. After finding an equilibrium point by the simplex method, we have used the information contained in the tableaux to get local optimality conditions for LBP. Based on these conditions, we have proposed a local algorithm which essentially comprises simplex pivot steps and thus can be implemented by using a simplex code.

One can easily see that the computational effort of the algortihm mostly depends on the number of linear programs $\min\{G_i^T s + M(\bar{z}^k)^T s : s \in S\}$ that are solved at Step 3 and the number of pivot steps demanded to find their solutions. The number of problems solved, which is at most $|N^+|$, can be reduced by first considering the directions $G_i$ which are not degenerate. Thus, as soon as an $i \in N^{+0}$ is found, the right decision of Step 3 can be accomplished. Of course, the use of Lemmas 15 and 21 also allows the reduction of the number of problems to be solved. Additionally, solving each problem from the optimal tableau of $P(M, \bar{s}^k)$ will take fewer pivot steps.

We have implemented the algorithm in C and run it with all the 384 test problems described in [16], which were randomly generated with a method based in [7]. The number of variables $n_1 + n_2$ varies from 50 up to 200 and the number of constraints $m$ is about 30% the number of variables. The percentage of variables attributed to the first level is 25% for half of the instances and 75% for the other half. The computational times spent on a AMD XP 1.7 GHz, with 512Kb of RAM, were negligible. For each problem, a local optimal solution was achieved in less than 1 millisecond. Let us note all test problems have a nondegenarate set $Z$ whereas degenerate vertices of set $S$ were generally reached before finding a local optimum.

Although the theoretical features and the preliminary computational results speak in favor of the algorithm and indicate that it can handle large LBPs, more computational experiments are needed to confirm the predicted performance. Specially, a careful computational experiment envolving primal degenerate test problems deserves to be carried out.

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