(Circular) backbone coloring of graphs

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I. Abstract

GRAPH COLORING problems are a class of well-known problems in GRAPH THEORY. In Telecommunications, GRAPH COLORING problems appear in the form of FREQUENCY ASSIGNMENT problems, where each frequency corresponds to a number in $\{1, \cdots, k\}$, named a *color*, and we want the radio-transmitters to not interfere with one another, usually considering an edge between those that are close and assigning different frequencies (i.e. different colors). Depending on the constraints on each situation, we may even want the difference between the frequencies to be greater than a fixed value for certain edges. This leads to several special cases in the GRAPH COLORING problems, of which we work with one named BACKBONE COLORING.

Given a graph $G = (V(G), E(G))$ and a subgraph $H = (V(H), E(H))$ of G, we say that (G, H) is a *pair* and the subgraph H is the *backbone* of G. Given integers $k \geq 1$ and $q \geq 2$, a *q-backbone k-coloring* ϕ of (G, H) is a proper coloring of G such that for every $uv \in E(H)$ we have $|\phi(u) - \phi(v)| \geq q$. The smallest value of k such that (G, H) is q-backbone k-colorable is named the *q-backbone chromatic number* of G and is denoted by $\mathrm{BBC}_q(G, H)$. Similarly, a *circular q-backbone k-coloring* ϕ of (G, H) is a proper coloring of G such that for every $uv \in E(H)$ we have $k - q \geq |\phi(u) - \phi(v)| \geq q$. The smallest value of k such that (G, H) is circular q-backbone k-colorable is named the *circular q-backbone chromatic number* of G and is denoted by $\mathrm{CBC}_q(G, H)$.

BACKBONE COLORING problems were first introduced by Broesrma et al. [1] in 2003. Since then, several other authors have worked in these problems. On our work, we have proved the following two theorems:

Theorem 1: Let G be a planar graph without cycles of length four and F be a forest of induced paths of G. The pair (G, F) is circular 2-backbone 7-colorable.

Theorem 2: Let G be a connected graph, $q \geq 2$ and $k \geq \max\left\{\chi(G), \left\lceil \frac{\chi(G)}{2} + q \right\rceil\right\}$. There exists a connected spanning subgraph H of G such that (G, H) is q-backbone k-colorable.

A. Regarding Theorem 1

Araújo et al. [2] proved a similar theorem, but in their work, they also forbid the graph G to have cycles of length five. The proof here is done by contradiction, where we assume the existence of pairs that do not satisfy the theorem. We study the structure of a *minimal pair*, that is, a pair (G, F) such that it cannot be circular 2-backbone 7-colored, but every proper subpair of (G, F) can. We show two inequalities that contradict each other and conclude that such minimal pair cannot exist.

For both inequalities we need to consider the influence of the cycles of length five. Consider a plane representation of G, we denote by *islands* the groups of faces of length at least five. A *bad island* is an island such that every face is a C_5 and that the graph induced by these faces in the dual graph of G form a tree. We obtain the first inequality simply by double counting edges in G, three for each triangle and one for each C_5 in G, and end up showing an upper bound for the number of edges in G

For the second inequality we use a minimal pair (G, F), remove certain vertices in F and show what we call *Reduction Propositions*. By induction we use these reductions to show that if the average degree of the vertices in the components of F is too small, then we can extend a coloring of a proper subpair of (G, F) to itself. Therefore we find a lower bound to the number of edges and this bound contradicts the first one, concluding that the minimal pair cannot exist.

B. Regarding Theorem 2

This second theorem generalizes the works of Bu et al. [3], [4], [5]. We prove it by contradiction. Consider a proper coloring ϕ of G. We denote $G_{\phi,q}$ to be the spanning subgraph of G such that for every edge $uv \in E(G)$ such that $|\phi(u) - \phi(v)| \geq q$, then $uv \in E(G_{\phi,q})$. Choose ϕ that maximizes the largest connected component of $G_{\phi,q}$, say H. Since G is connected, there exist edges leaving H and into $G - H$. The main idea is to change the color of an endpoint v of one of these edges in a way that we obtain a coloring φ such that the largest connected component of $G_{\varphi,q}$ has more vertices than the one in $G_{\phi,q}$, which is a contradiction by our choice of ϕ.

References

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