Decycling with a matching

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I. Introduction

Given a finite, simple, and undirected graph of order n and size m, destroying all cycles by removing vertices or edges is a classical theme. The minimum number of edges whose removal destroys all cycles is exactly $m - n + 1$, and standard minimum spanning tree algorithms allow to solve even weighted optimization versions. Contrary to this, the minimum number of vertices whose removal yields a tree is a difficult parameter [1]–[3], [5], [6].

We study the apparently simple case when the removed edges are required to form a matching. Quite surprisingly, we show that the corresponding decision problem, that is, the problem to decide whether a given graph is the union of a tree and a matching, is already hard. Furthermore, we present efficient algorithms for a number of well-known graph classes.

For a set E of edges of a graph G, let $G - E$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E$. If $G - E$ is a forest, then E is *decycling*. Let $\mathcal{FM}$ be the set of all graphs that have a decycling matching.

II. Results

The following lemma collects some basic observations concerning graphs that have a decycling matching.

***Lemma** 1:* Let G be a graph.

(i) If $G \in \mathcal{FM}$ is connected, then G has a matching M for which $G - M$ is a tree.
(ii) If $G \in \mathcal{FM}$, then $m(H) \leq \left\lfloor \frac{3n(H)}{2} \right\rfloor - 1$ for every subgraph H of G.
(iii) If G is subcubic and connected, then $G \in \mathcal{FM}$ if and only if G has a spanning tree T such that all endvertices of T are of degree at most 2 in G.

Lemma 1(iii) is the key observation for the following result.

***Theorem** 2:* For a given 2-connected planar subcubic graph G, it is NP-complete to decide whether $G \in \mathcal{FM}$.

Sketch of Proof: The considered decision problem is clearly in NP. The 3-connected planar cubic graphs G constructed in [4] contain several edges that necessarily belong to every Hamiltonian cycle of G; regardless of whether such a cycle exists or not. Therefore, removing such an edge, their construction implies the NP-completeness of the following decision problem: *Given a 2-connected planar subcubic graph G with exactly two vertices u and v of degree 2, does G have a Hamiltonian path whose endvertices are u and v?*

Let G be a 2-connected planar subcubic graph with exactly two vertices u and v of degree 2. In order to complete the proof, it suffices to show that G has a Hamiltonian path whose endvertices are u and v if and only if $G \in \mathcal{FM}$. ■

We also consider a more general decision problem.

Allowed Decycling Matching

Instance:	A graph G and a set F of edges of G.
Task:	Decide whether G has a decycling matching M that does not intersect F, and determine such a matching if it exists.

For this new version, we summarize our positive results as follows.

***Theorem** 3:* Allowed Decycling Matching can be solved in polynomial time for $\{K_{1,3}, K_{1,3} + e\}$-free graphs.

***Theorem** 4:* Allowed Decycling Matching can be solved in polynomial time for P_5-free graphs.

***Theorem** 5:* Allowed Decycling Matching can be solved in polynomial time for chordal graphs.

***Theorem** 6:* Allowed Decycling Matching can be solved in polynomial time for C_4-free distance hereditary graphs.

III. Conclusion

We study a special case of the more general problem of destroying all cycles by removing edges under the restriction that the graph formed by the removed edges has bounded maximum degree. This problem can certainly be considered more generally. Furthermore, one can consider variants, such as, for instance, deciding whether a given graph is the union of a bipartite graph and a matching, that is, whether it has a matching whose removal destroys all odd cycles.

References

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