Two related questions on total coloring of cubic graphs

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***Abstract*—By proposing two questions on total colorings of cubic graphs of large girth, we investigate a possible connection between girth and total chromatic parameters in cubic graphs.**

I. Introduction

A k-total-coloring of a graph G is an assignment of k colors to the edges and vertices of G, so that adjacent and incident elements have different colors. The total chromatic number of G is the least k for which G has a k-total-coloring. The Total Coloring Conjecture states that the total chromatic number of any graph is at most $\Delta+2$ [1], [14]. This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (called Type 1) or 5 (called Type 2) [11], [13], see also [7] for a recent concise proof. It is NP-hard to decide whether a cubic graph is Type 1, even restricted to bipartite cubic graphs [10].

The Type of all cubic graphs with order up to 16 is established [4], [9] as well as the Type of infinite families of cubic graphs [4], [12], [3]. So far every known Type 2 cubic graph contains a square or a triangle, and computational results show that a possible Type 2 cubic graph with girth greater than 4 must have at least 34 vertices [2]. In this context, we proposed the following question.

Question 1 (Brinkmann et al. [2]): Does there exist a Type 2 cubic graph with girth greater than 4?

The equitable total coloring requires further that the cardinalities of any two color classes differ by at most 1. Similarly to the situation with total colorings, it was conjectured that the equitable total chromatic number of any graph is at most $\Delta + 2$, and this conjecture was proved for cubic graphs in the same work [15]. We showed in [6] that it is NP-complete to decide whether the equitable total chromatic number of a bipartite cubic graph is equal to 4.

Also in [6], we established for one infinite family of Type 1 cubic graphs of girth 5 that they all have equitable total chromatic number 4. Furthermore, we presented the first known Type 1 cubic graphs with equitable total chromatic number 5. All of them have, by construction, girth either 4 or 3. This is maybe related to Question 1. Indeed, the high constraint on the 4-total-colorings of the graph used in this construction is the reason why $K_{3,3}$ is Type 2 (the graph is obtained from $K_{2,3}$ by adding one pendant edge incident to each vertex of degree 2 of $K_{2,3}$). There are probably other Type 2 cubic graphs that could provide graphs with the same properties as that one and that could be used for the construction. However, since no Type 2 cubic graph of girth at least 5 is known, we were led to the following question.

Question 2 (Dantas et al. [5]): Does there exist a Type 1 cubic graph with girth greater than 4 and equitable total chromatic number 5?

Remark: There are many other questions that we could not answer in this study, besides these two questions presented here. For instance, another question proposed in [2] is

Is there a girth g such that all cubic graphs with girth at least g are Type 1?

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