# Approximation Algorithms for the Max-Buying Problem with Limited Supply*

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## 1. Introduction

An interesting economic problem faced by companies is to choose the price of items to be sold in order to maximize revenue. One way to address this problem is through the non-parametric approach [1], where, given a set $I$ of $n$ items and $B$ of consumers and values $v_{ib}$ for every item $i$ and consumer $b$ indicating the largest amount that a consumer $b$ is willing to pay for item $i$, the seller wants to assign a price $p_i$ for each item $i$ with the objective of maximizing the revenue considering that a consumer $b$ will only buy a *feasible* item, that is, an item such that $p_i \leq v_{ib}$.

Among the models in the literature, we have the *Max-Buying Problem* (MAX), where every consumer buys one feasible item and the *Rank-Buying Problem* (RANK), where every consumer has a preference order on the item and her/him will buy the most preferred feasible item. For this problem, we also consider *consistent budgets*, that is, if a consumer $b$ prefers item $i$ to item $j$, then $v_{ib} \geq v_{jb}$. One can consider that there is an *unlimited supply* of every item, that is, an item can be sold multiple times or that every item $i$ has a maximum number of copies $C_i$ that can be sold (*limited supply*). Also, sometimes a company knows (or desires) an ordering in the prices of the items, so one can require a *price ladder*, that is, one is interested only in pricings such that $p_1 \geq p_2 \geq \cdots \geq p_n$.

## 2. Max-Buying Problem with Limited Supply

For an item $i$, let $\mathcal{S}(i) = \{(i, S) \colon S \subseteq B,\, |S| \leq C_i\}$. We call a pair $(i, S)$ in $\mathcal{S}(i)$ a *star* of $i$ and we denote $\bigcup_{i \in I} \mathcal{S}(i)$ by $\mathcal{S}$. A feasible solution of MAX can be seen as a collection of stars, one for each item, with every consumer in at most one star, and the price of an item $i$ being $P_{(i,S)} = \min\{v_{ib} \colon b \in S\}$, where $(i, S)$ is the star of item $i$ in the collection, leading to the following formulation:

$$
\begin{array}{llll}
\text{(SF)} & \max & \sum_{(i,S)\in\mathcal{S}} |S| P_{(i,S)} x_{(i,S)} & \\
 & \text{s.t.} & \sum_{(i,S)\in\mathcal{S}(i)} x_{(i,S)} = 1, & \forall i \in I \\
 & & \sum_{(i,S)\in\mathcal{S}: b\in S} x_{(i,S)} \leq 1, & \forall b \in B \\
 & & x_{(i,S)} \in \{0,1\}, & \forall (i,S) \in \mathcal{S}.
\end{array}
$$

where $x_{(i,S)}$ indicates if item $i$ is sold to the set $S$ of consumers. Even though (SF) can have an exponential number of variables, its linear relaxation $(\mathrm{SF}_R)$ can be solved in polynomial time [3]. Thus, we can consider the following algorithm.

STARROUNDING:

1) Find an optimal solution $x$ of $(\mathrm{SF}_R)$
2) For every item $i$, choose a star $S_i \in \mathcal{S}(i)$ with probability $\mathbb{P}(S_i = (i, S)) = x_{(i,S)}$
3) For every consumer $b$, sell to $b$ the item $i$ to such $P_{S_i}$ is maximum with $S_i = (i, S)$ and $b \in S$

**Theorem 1** (Fernandes and Schouery [3]). STARROUNDING *is an $e/(e-1)$-approximation for* MAX *with limited supply.*

Also, using dynamic programming, it is possible to develop an algorithm for the case with a price ladder.

**Theorem 2** (Fernandes and Schouery [3]). *For every $0 < \varepsilon < 1$, there is a $(2+\varepsilon)$-appro-ximation for* MAX *with limited supply and a price ladder.*

## 3. Open Problems

Among the open problems regarding non-parametric pricing problems, we have the problem of developing an algorithm with approximation ratio better than $e/(e-1)$ for MAX with limited [3] or unlimited [1] supply; the problem of developing an algorithm with approximation ratio better than $2+\varepsilon$ for MAX with limited supply and a price ladder; and the problem of developing an approximation algorithm for RANK without a price ladder or with a price ladder and limited supply. Currently, it is only known a PTAS for RANK with unlimited supply and a price ladder [1]. RANK is strongly NP-hard [2] and, without consistent budget, it is not in APX unless $\mathrm{P} = \mathrm{NP}$ [2].

## References

[1] G. Aggarwal, T. Feder, R. Motwani, and A. Zhu. Algorithms for multi-product pricing. In *ICALP'04*, pages 72–83, 2004.

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