A proof for a Conjecture of Gorgol

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The first known result in extremal graph theory was published in 1907 when Mantel [1] proved that the maximum number of edges in a simple graph without a triangle as a subgraph is $\lfloor n^2/4 \rfloor$. Turán [2] generalized this result to find the maximum number of edges without K_r as a subgraph, where K_r denotes the complete graph on r vertices.

The *Turán number* $\mathrm{ex}(n, H)$ is the maximum number of edges in a graph on n vertices which does not contain H as a subgraph. Using this notation, Mantel's Theorem states that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. Let $H_{\mathrm{ex}}(n, G)$ represent an *extremal graph* on n vertices without G as a subgraph with $\mathrm{ex}(n, G)$ edges. Turán's Theorem states that $H_{\mathrm{ex}}(n, K_{r+1}) = T_r(n)$, where $T_r(n)$ is the complete r-partite graph on n vertices in which all parts have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

For a general graph H, the value of $\mathrm{ex}(n, H)$ is asymptotically well known. Erdős, Stone and Simonovits' Theorem [3] states that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{n^2} = \frac{\chi(H) - 2}{2\chi(H) - 2}.$$

Although this theorem gives a lot of information on the asymptotic growth of $\mathrm{ex}(n, H)$, it should be noted that it is only of interest for nonbipartite graphs. If H is bipartite, it asserts merely that $\mathrm{ex}(n, H) \ll n^2$. In this paper, we focus on the Turán number for bipartite graphs. In particular, we consider when $H = kP_3$, where P_r is a path on r vertices and kG consists of k disjoint copies of the graph G.

For graphs G and H, let $G + H$ and $G \vee H$ be the disjoint union and join of G and H, respectively. The join of G and H is the graph obtained from $G + H$ by adding edges from all vertices of G to all vertices of H. Simonovits [4] showed that the graph $K_{k-1} \vee T_r(n - k + 1)$ is the unique extremal graph forbidding kK_{r+1} for sufficiently large n. Gorgol [5] gave upper and lower bounds for the Turán numbers forbidding kH, for a connected graph H on r vertices. The lower bound was obtained by noting that neither $G_1(n, kH) = K_{kr-1} + H_{\mathrm{ex}}(n-kr+1, H)$ nor $G_2(n, kH) = K_{k-1} \vee H_{\mathrm{ex}}(n-k+1, H)$ contain k disjoint copies of H. Indeed, for $G_1(n, kH)$, K_{kr-1} does not have enough vertices to contain kH and $H_{\mathrm{ex}}(n - kr + 1, H)$ contains no copies of H. In the second case, since $H_{\mathrm{ex}}(n - k + 1, H)$ contains no copies of H, any copy of H in $G_2(n, kH)$ must contain at least one vertex in K_{k-1} which implies that there can be at most $k - 1$ copies of H in $G_2(n, kH)$. Gorgol's lower bound is, therefore,

$$\mathrm{ex}(n, kH) \geq \max\{\mathrm{e}(G_1(n, kH)), \mathrm{e}(G_2(n, kH))\}$$

where $\mathrm{e}(G)$ denotes the number of edges in G.

We consider the case when $H = P_3$. Let $\mathrm{g}_1(n, k) = \mathrm{e}(G_1(n, kP_3))$, $\mathrm{g}_2(n, k) = \mathrm{e}(G_2(n, kP_3))$ and $\mathrm{Gorgol}(n, k) = \max\{\mathrm{g}_1(n, k), \mathrm{g}_2(n, k)\}$. Note that $H_{\mathrm{ex}}(n, P_3) = M_n$, where M_n is a *near perfect matching* on n vertices. If n is even, then M_n consists of a matching on n vertices and, if n is odd, then $M_n = M_{n-1} + K_1$. Thus, $\mathrm{e}(M_n) = \lfloor n/2 \rfloor$ and we have

$$\mathrm{g}_1(n, k) = \binom{3k-1}{2} + \left\lfloor \frac{n - 3k + 1}{2} \right\rfloor;$$

$$\mathrm{g}_2(n, k) = \binom{k-1}{2} + (n - k + 1)(k - 1) + \left\lfloor \frac{n - k + 1}{2} \right\rfloor;$$

and

$$\mathrm{Gorgol}(n, k) = \begin{cases} \mathrm{g}_1(n, k), & \text{for } 3k \leq n < 5k, \\ \mathrm{g}_2(n, k), & \text{for } n \geq 5k. \end{cases}$$

When $n < 3k$, a graph on n vertices cannot contain kP_3 as a subgraph. Gorgol's conjecture states that $\mathrm{ex}(n, kP_3) = \mathrm{Gorgol}(n, k)$ when $n \geq 3k$ and, in the same paper, Gorgol proved this is true when $k \in \{2, 3\}$. Bushaw and Kettle [6] also proved the conjecture to be true for $n \geq 7k$.

In this paper, we give a constructive proof for Gorgol's Conjecture for all values of n and k. In particular, we give an algorithm that finds a set of disjoint copies of P_3 given a graph G as input. We prove Gorgol's Conjecture by showing that, if G has n vertices and more than $\mathrm{Gorgol}(n, k)$ edges, then this algorithm returns at least k disjoint copies of P_3 in G.

References

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