Tractability, Hardness, and Kernelization Lower Bound for And/Or Graph Solution

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I. Introduction

And/or graphs are a well-known data structure with several applications in many fields of computer science, such as Artificial Intelligence, Distributed Systems, Software Engineering, and Operations Research. Such structure consists of an acyclic digraph G containing a single source vertex s, where every vertex $v \in V(G)$ has a label $f(v) \in \{\texttt{and}, \texttt{or}\}$. In an and/or graph, (weighted) edges represent dependency relations between vertices: a vertex labeled and depends on all of its out-neighbors, while a vertex labeled or depends on only one of its out-neighbors. A solution subgraph H of an and/or graph G is a subdigraph of G containing its source vertex and such that if an and-vertex (resp. or-vertex) is included in H then all (resp. one) of its out-edges must also be included in H. In general, solution subgraphs represent viable solutions of problems modeled by and/or graphs. The optimization problem associated with an and/or graph G consists of finding a *minimum weight solution subgraph* H of G, where the weight of a solution subgraph is the sum of the weights of its edges. Given its wide applicability, in this work we develop a multivariate investigation into this optimization problem. In a previous paper [U. S. Souza, F. Protti, M. Dantas da Silva, Revisiting the complexity of and/or graph solution, J. Comput. Syst. Sci. 79:7 (2013) 1156-1163] we have analyzed the complexity of such a problem under various aspects, including parameterized versions of it. However, the main open question has remained open: Is the problem of finding a solution subgraph of weight at most k (where k is a fixed parameter) in FPT? In this paper we answer negatively to this question, proving the W[1]-hardness of the problem, and its W[P]-completeness when zero-weight edges are allowed. We also show that the problem is fixed-parameter tractable when aggregated to k a natural parameter of and/or graphs, very important in practice; and remains fixed-parameter tractable when parameterized by tree-width. Finally, using a framework developed by Bodlaender *et al*. (2009) and Fortnow and Santhanam (2011), based upon the notion of compositionality, we show that the tractable cases do not admit a polynomial kernel unless $NP \subseteq coNP/poly$, even restricted to instances without or-vertices with out-degree greater than two.

The main optimization problems associated with and/or graphs is formally defined below.

Min-and/or
Instance: An and/or graph $G = (V, E)$, containing a single source vertex s, and such that each edge e of G has an integer weight $\tau(e) > 0$.
Goal: Determine the minimum weight of a subdigraph $H = (V', E')$ of G (*solution subgraph*) satisfying the following properties:

- $s \in V'$;
- if a vertex v is in V' and $f(v)$=and then every out-edge of v belongs to E';
- if a non-sink vertex v is in V' and $f(v)$=or then exactly one out-edge of v belongs to E'.

Min-and/or^0 is a generalization of Min-and/or where zero-weight edges are allowed.

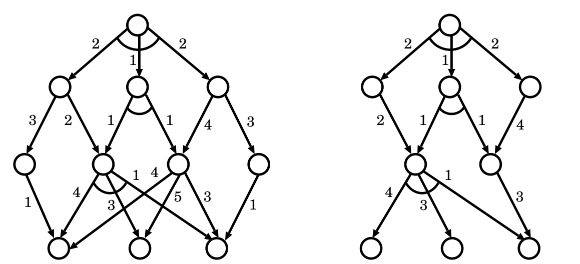

Fig. 1. (a) An and/or graph G; (b) An optimal solution subgraph of G.

In 1974, Sahni showed that Min-and/or is NP-hard via a reduction from 3-Sat. We denote by Min-and/or(k) the parameterized version of Min-and/or which asks whether there is a solution subgraph of weight at most k. This approach is justified by the fact that many applications are concerned with satisfying a low cost limit. The question "Min-and/or$(k) \in$ FPT?" has remained open up to now. In this paper we close this question by proving that Min-and/or(k) is W[1]-hard and in $W[P]$; we also show that Min-and/or$^0(k)$ is in fact W[P]-complete. In addition, some fixed-parameter tractable cases are shown, and using a framework developed by Bodlaender *et al.* and Fortnow and Santhanam, which is based on the notion of *compositionality*, we show that such tractable cases do not admit a polynomial kernel unless $NP \subseteq coNP/poly$. The latter condition, if true, would imply an unlikely collapse of the polynomial hierarchy to the third level ($PH = \Sigma_p^3$).

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