STIC-AmSud Kickoff meeting – Group 2 summary
Polyhedral Studies on vertex coloring problems

Background

Given a graph $G = (V, E)$, a *vertex coloring* of G is an assignment $c : V \to \mathbb{N}$ of "colors" to vertices of G, such that $c(v) \neq c(w)$ for each edge $vw \in E$. The *classical vertex coloring problem* consists in finding a coloring of G minimizing the number of required colors. This parameter is widely known as the *chromatic number* of G, and is denoted as $\chi(G)$. There are many variants for the graph coloring problem, some of them being: Precoloring extension [1], μ-coloring [2], (γ, μ)-coloring [3], List coloring [14] and many other variants.

Although the classical vertex coloring problem is NP-hard [9], there are many graph classes for which this problem can be solved in polynomial time, one of the most important classes being perfect graphs [10]. However, the variants of the coloring problem mentioned above may not be polynomially solvable on these graphs. On [3, 4], the complexity boundary between coloring and list-coloring is studied for several subclasses of perfect graphs.

In the last decade, *Integer linear programming* (ILP) has been succesfully applied to graph coloring problems, by resorting to several formulations for the classical version of the problem. The *Standard formulation* uses a binary variable x_{ic} for each vertex $i \in V$ and each color c to indicate wether vertex i is assigned color c or not. There are many other known formulations such as the *Orientation model* [5], the *Distance model* [8], the *Representatives formulation* [6, 7] and the *Maximal Independant Sets formulation* [12], among others.

In general, solving an ILP is NP-hard. However, in many cases, a complete description of the convex hull of its solutions is known and this description can be used to solve the *separation problem* for the polytope associated to the formulation in polynomial time [13]. Based on the ellipsoid method, Gröschel, Lovász and Schrijver [11] proved that the separation problem and the optimization problem over a polytope are polynomially equivalent. From this equivalence, there exists a generalized conjecture suggesting that if a combinatorial optimization problem can be solved in polynomial time, then there should exist some ILP formulation of the problem for which the convex hull of its solutions admits an "elegant" characterization; formally, for which the separation problem over this polytope can be solved in polynomial time.

Despite the fact that many vertex coloring problems are polynomially solvable on certain graph classes, most of these problems are not "under control" from a polyhedral standpoint. The mentioned equivalence between optimization and separation *suggests* that, for these problems, there must exist formulations admiting some elegant characterization for the polytopes associated to them, therefore it is interesting to study the mentioned formulations (if not others) with the goal of finding such characterizations for these problems.

In this sense, some work has been done on the Standard formulation and nice characterizations were found for some simple graph classes such as trees and block graphs. Also, some general results imply that the separation problem associated to this formulation cannot be solved in polynomial time when the list coloring problem on the family analyzed is NP-complete, meaning that this formulation falls short too soon. Therefore, one direct next step on this line of work would be to study these families via other formulation.

The Assymetric Representatives Formulation [7]

Given a graph $G = (V, E)$ and an order $\prec$ on the set of vertices, the set of anti-neighbours of a vertex $u \in V$, $\bar{N}(u) = \{v \in V \ / \ uv \notin E\}$, can be splitted into its *negative anti-neighbourhood* $\bar{N}^-(u) = \{v \in \bar{N}(u) \ / \ v \prec u\}$ and its *positive anti-neighbourhood* $\bar{N}^+(u) = \{v \in \bar{N}(u) \ / \ u \prec v\}$. In the Assymetric Representatives Formulation for the classical vertex coloring problem, for each $u \in V$ there is a binary variable x_{uu} stating whether u is the representative of its own color class or not. Additionally, for each $v \in \bar{N}^+(u)$, a binary variable x_{uv} states whether vertex u is the representative of the color class assigned to vertex v or not. With these definitions, a vector satisfying

$$x_{uu} + \sum_{v \in \bar{N}^-(u)} x_{vu} = 1 \qquad \forall u \in V \tag{1}$$

$$\sum_{v \in K} x_{uv} \leq x_{uu} \qquad \forall u \in V, \ \forall \text{ clique } K \subseteq \bar{N}^+(u) \tag{2}$$

$$x_{uu}, x_{uv} \in \{0, 1\} \qquad \forall u \in V, \ \forall v \in \bar{N}^+(u),$$

represents a proper coloring of G.

In the above formulation, every variable x_{uu} can be cleared from equation (1) to be

$$x_{uu} = 1 - \sum_{v \in \bar{N}^-(u)} x_{vu}.$$

Hence, these variables can be eliminated from the formulation rewriting constraints (2) as

$$\sum_{v \in \bar{N}^-(u)} x_{vu} + \sum_{v \in K} x_{uv} \leq 1 \qquad \forall u \in V, \ \forall \text{ clique } K \subseteq \bar{N}^+(u) \tag{3}$$

It is easy to see that the polytope described by (3) is the Stable Set Polytope on some special graph $H(G)$. This graph $H(G)$, has one vertex for each edge of G, and two vertices are adjacent in $H(G)$ if the corresponding edges in G appear in the same constraint from (3). Therefore, a complete characterization of $STAB(H(G))$ would yield a complete characterization of the vertex coloring polytope associated to the representatives formulation for the original graph G.

Since the stable set polytope has been widely and deeply studied, an interesting approach on finding nice characterizations for vertex coloring polytopes associated to some graph families would be to study the graphs obtained from these families, as described above.

Proposed lines of work and main objectives

Given a family $\mathcal{G}$ of graphs, we want to study the family $\mathcal{H}(\mathcal{G}) = \{H(G) \ / \ G \in \mathcal{G}\}$, since a complete characterization of the Stable Set polytope associated to $\mathcal{H}(\mathcal{G})$, $STAB(\mathcal{H}(\mathcal{G}))$, will give us a complete characterization of the vertex coloring polytope associated to $\mathcal{G}$, $VCOL(\mathcal{G})$. After obtaining some characterization for $VCOL(\mathcal{G})$, it will be interesting to "reinterpret" the valid inequalities for $STAB(\mathcal{H}(\mathcal{G}))$ but now in the context of the vertex coloring and the original meaning of the variables.

Following this line of work, as a first result we proved that if $\mathcal{G}$ are the complements of triangle-free graphs, then $\mathcal{H}(\mathcal{G})$ are line graphs of $\mathcal{G}$. As the $STAB$ for line graphs is completely characterized, we have now a complete characterization for $VCOL$ for complements of triangle-free graphs. An interesting approach would be to detect those graph families $\mathcal{G}$ for wich $STAB(\mathcal{H}(\mathcal{G}))$ is known, such as perfect graphs. Yet if $STAB(\mathcal{H}(\mathcal{G}))$ is not already known but widely studied, the latter would be also interesting, e.g., if $\mathcal{H}(\mathcal{G})$ are claw-free graphs, as the general intuiton is that a characterization for this $STAB$ will appear sooner or later.

A direct polyhedral study over $VCOL(\mathcal{G})$ would be also an interesting line of work. In this sense, one (unchecked yet) first result imply that the separation/optimization problem over $VCOL(\mathcal{G})$ can be solved in polynomial time if and only if the same occurs for the Precoloring Extension polytope associated to $\mathcal{G}$[1]. This result will imply that if Precoloring Extension over $\mathcal{G}$ is an NP-complete problem, then the separation problem over $VCOL(\mathcal{G})$ cannot be solved in polynomial time and this would narrow the set of candidate families to study. A direct consequence of this result among the one in the previous paragraph is that the Precoloring Extension problem on complements of triangle-free graphs can be solved in polynomial time.

Another proposed line of work is to use some other methods on the construction of $H(G)$. It may be possible to derive another graph from G such as some polytope on this other graph is completely characterized.

Referencias

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[1] Sketch of the proof: the Precoloring Extension polytope can be obtained by reordering the vertices to start with the precolored ones and fixing some representative on each precolored class. As we have a {0,1}-polytope, the resulting polytope via this fixings is known, if the original polytope is also known.