# Polyhedral studies of vertex coloring polytopes

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# Outline

1. Introduction
   - Vertex coloring problems
   - Integer Programming models
   - Geometric algorithms (and implications...)
2. Standard IP formulation
   - The formulation and some general results
   - Trees, blocks and cacti graphs
3. Some final remarks and future work

# Outline

# Classical vertex coloring problem and variants

Given a graph $G = (V, E)$, find an assignment $c : V \to \mathbb{N}$ such that $c(v) \neq c(w)$ for every $vw \in E$.

Other vertex coloring problems:

- Pre-coloring extension: some vertices $v \in V$ are *pre-colored* (i.e., $c(v)$ is fixed for these vertices).
- $\mu$-coloring: each vertex has an upper bound, $\mu(v)$, for its assigned color (i.e., $c(v)$ must be at most $\mu(v)$).
- $(\gamma, \mu)$-coloring: in addition to $\mu(v)$, each vertex has a lower bound, $\gamma(v)$, for its color (i.e., $c(v)$ must be at least $\gamma(v)$).
- List-coloring: each vertex has a list, $L(v)$, of possible colors for it (i.e., $c(v)$ must belong to $L(v)$).

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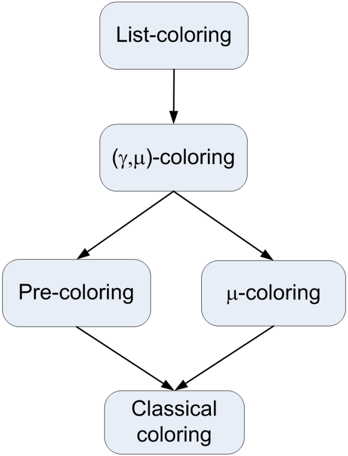
# Classical vertex coloring problem and variants

These problems can be arranged in the following complexity hierarchy:



# Classical vertex coloring problem and variants

Complexity boundary for some graph families:

| Class | Coloring | Pre-col | $\mu$-col | $(\gamma, \mu)$-col | List-col |
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| Complete bipartite | P | P | P | P | NP-c |
| Bipartite | P | NP-c | NP-c | NP-c | NP-c |
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| Complements of bipartites | P | P | ? | ? | NP-c |
| Trees | P | P | P | P | P |
| Block | P | P | P | P | P |
| Cacti | P | P | P | P | P |

"NP-c": NP-complete problem "P": polynomial problem "?": open problem

# Outline

# Standard IP formulation

(Coll et al., 2002; Mendez Díaz & Zabala, 2006, 2008)

Given a graph $G = (V, E)$ and a set of colors $C$, the standard IP formulation for vertex coloring problems uses a binary variable $x_{ic}$ for every vertex $i \in V$ and every color $c \in C$ subject to the following constraints:

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\begin{aligned}
\sum_{c \in C} x_{ic} &= 1 && \forall i \in V \\
x_{ic} + x_{jc} &\leq 1 && \forall ij \in E, \forall c \in C \\
x_{ic} &\in \{0, 1\} && \forall i \in V, \forall c \in C
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**Observation**: this formulation can be extended with a binary variable $w_c$ for each color $c \in C$ indicating whether $c$ is used in the assignment or not.

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# Orientation model
(Borndörfer et al., 1998)

For $i \in V$, an integer variable $x_i$ is used to represent the color assigned to $i$. It introduces a binary orientation variable $y_{ij}$ for each $ij \in E$ such that $y_{ij} = 1$ if and only if $x_i < x_j$.

$$
\begin{aligned}
y_{ij} + y_{ji} &= 1 && \forall ij \in E, i < j \\
x_i - x_j &\geq 1 - (|V| + 1) y_{ij} && \forall ij \in E, i < j \\
-x_i + x_j &\geq 1 - (|V| + 1) y_{ji} && \forall ij \in E, i < j \\
x_i &\in \mathbb{Z} && \forall i \in V \\
y_{ij} &\in \{0, 1\} && \forall ij \in E.
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# Distance model
(D. & Marenco, 2009)

For each pair of vertices $i, j \in V$ with $i < j$, an integer variable $x_{ij}$ determines the *distance* between the assigned to $i$ and $j$. Orientation binary variables $y_{ij}$ are also used as in the previous orientation model.

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\begin{aligned}
y_{ij} + y_{ji} &= 1 & \forall ij \in E, i < j \\
x_{ij} &= x_{ik} + x_{kj} & \forall i, k, j \in V, i < k < j \quad (!) \\
-(|C| - 1) \leq x_{ij} &\leq |C| - 1 & \forall i, j \in V, i < j \\
x_{ij} &\geq 1 - |C| y_{ij} & \forall ij \in E, i < j \\
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# Maximal Independent Sets (MIS) model
(Mehrotra & Trick, 1996)

For each maximal stable set $S \subseteq V$, a binary variable $x_S$ is used to determine if $S$ is used in the coloring, i.e., $S$ represents a color class.

$$\begin{aligned} \sum_{S: i \in S} x_S &\geq 1 && \forall i \in V \\ x_S &\in \{0, 1\} && \forall S \in \mathcal{S}(G) \end{aligned}$$

where $\mathcal{S}(G)$ contains every maximal stable set of $G$.

# Representatives model
(Campêlo et al., 2004)

For $i \in V$ and $j \in \bar{N}[v]$, a binary variable $x_{ij}$ determines if $i$ is the *representative* of the color class assigned to $j$.

$$\begin{array}{rcll} \sum_{i \in \bar{N}[j]} x_{ij} & \geq & 1 & \forall j \in V \\ x_{ij} + x_{ik} & \leq & x_{ii} & \forall i \in V,\ \forall jk \in E : j,k \in \bar{N}(i) \\ x_{ij} & \in & \{0,1\} & \forall i \in V,\ \forall j \in \bar{N}[i]. \end{array}$$

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# Optimization and separation problems

Let $P$ be a convex and compact set in $\mathbb{R}^n$. The following are two well-known algorithmic problems in connection with $P$:

**Definition (Optimization problem)**

Given a vector $c \in \mathbb{R}^n$, find a vector $y$ that maximizes $c^T x$ on $P$, or assert that $P$ is empty.

**Definition (Separation problem)**

Given a vector $y \in \mathbb{R}^n$, decide whether $y \in P$ and, if not, find a hyperplane that separates $y$ from $P$; i.e., find a vector $c \in \mathbb{R}^n$ such that $c^T y > max\{c^T x : x \in P\}$.

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# Optimization and separation problems

In 1981, Gröschel, Lovász and Schrijver proved a fundamental theorem for polyhedral theory:

**Theorem (Gröschel, Lovász and Schrijver, 1981)**

*Given a convex and compact set $P \in \mathbb{R}^n$, the optimization problem over P can be solved in polynomial time if and only if the separation problem over P can be solved in polynomial time.*

# The generalized conjecture

This result leads to the following (generalized) conjecture:

**Conjecture**

Given a problem $\mathcal{P}$, if there exists a polynomial time algorithm to solve it, then there is a "decent" linear programming model describing the feasible solutions of $\mathcal{P}$.

To find such characterizations for graph coloring problems is the main objective of our work!

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# Motivation

## Why?

- Theorethical: To complete the polyhedral counterpart of combinatorially-solved graph coloring problems.
- Practical: Studying these polytopes may lead us to (polyhedrally) solve some other open problems.
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# Some definitions first...

**Definition (Standard coloring polytope)**

Given a graph $G$ and a set of colors $C$, we define $\mathcal{P}(G, C)$ to be the convex hull of the incident vectors of C-colorings of $G$.

**Definition (Standard list-coloring polytope)**

Given a graph $G = (V, E)$, a set of colors $C$ and a set $L$ of lists $L(i)$, for $i \in V$ of possible assignments for the vertices of $G$, we define $\mathcal{PL}(G, C, L)$ to be the convex hull of the incident vectors of (C,L)-list colorings of $G$.

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# Some general results

## Theorem

*Given a graph $G$ and a set of colors $C$, the **separation** problem over $\mathcal{P}(G, C)$ can be solved in polynomial time **if and only if** the **separation** problem over $\mathcal{PL}(G, C, L)$ can be solved in polynomial time for any list $L$.*

### Sketch of the proof.

As $\mathcal{P}(G, C) \subseteq [0, 1]^{|V| \cdot |C|}$, it's easy to show that

$$\mathcal{PL}(G, C, L) = \mathcal{P}(G, C) \cap \{x : \sum_{c \notin L(i)} x_{ic} = 0\}.$$

Then, a point $\hat{x} \notin \mathcal{PL}(G, C, L)$ either does not belong to $\mathcal{P}(G, C)$ or has $\hat{x}_{ic} > 0$ sor some $c \notin L(i)$. Hence, to separate a point from $\mathcal{PL}(G, C, L)$ we just need to test if $\hat{x}_{ic} = 0$ sor all $c \notin L(i)$, or eventually separate the point (in polynomial time) from $\mathcal{P}(G, C)$. The converse is trivial. □

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# Some general results

The following is a direct consequence of GLS theorem:

**Corollary**

Given a graph $G$ and a set of colors $C$, the **optimization** problem over $\mathcal{P}(G, C)$ can be solved in polynomial time **if and only if** the **optimization** problem over $\mathcal{PL}(G, C, L)$ can be solved in polynomial time for any list $L$.

# Some general results

## Theorem

*Let $\mathcal{G}$ be a family of graphs and $C$ a set of colors. If the list-coloring problem on $(\mathcal{G}, C)$ is an NP-C problem, then the optimization/separation problem over the standard coloring polytope $\mathcal{P}(\mathcal{G}, C)$ cannot be solved in polynomial time, unless $P = NP$.*

## Proof.

If the optimization problem over $\mathcal{P}(\mathcal{G}, C)$ can be polynomially solved, then we can optimize over $\mathcal{PL}(\mathcal{G}, C, L)$, for any set of lists $L$, in polynomial time solving the list-coloring problem on $(\mathcal{G}, C)$, thus contradicting the hypothesis. □

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# Some general results

Some remarks about the standard formulation:

- This is a very simple, easy to study, formulation and it yields polytopes with strong combinatorial properties.
- Unfortunately, the above results show that this formulation is not very powerful, as its polytopes do not admit “nice” characterizations for hard problems.
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Recall the known complexity boundaries

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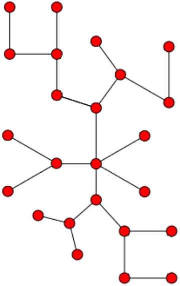
"NP-c": NP-complete problem "P": polynomial problem "?": open problem

# Outline

# Standard polytope on trees

Some insights on the vertex coloring problem and trees:

- Vertex coloring problems seem to be hard on cliques, holes and anti-holes.
- Trees do not contain any of these structures.



**Question**: If $G$ is a tree, do we need anything else but the standard model to describe $\mathcal{P}(G, C)$?
(i.e., is the linear relaxation of the model an integer polytope?)

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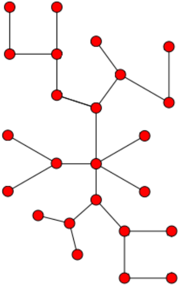

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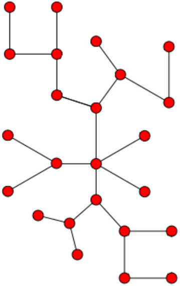

Some insights on the vertex coloring problem and trees:

- Vertex coloring problems seem to be hard on cliques, holes and anti-holes.
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**Question**: If $G$ is a tree, do we need anything else but the standard model to describe $\mathcal{P}(G, C)$?
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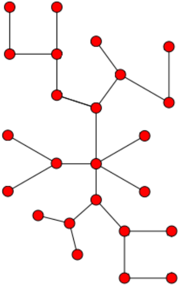
# Standard polytope on trees

Some insights on the vertex coloring problem and trees:

- Vertex coloring problems seem to be hard on cliques, holes and anti-holes.
- Trees do not contain any of these structures.



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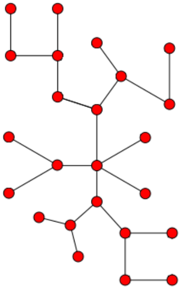

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**Theorem**

*Given a tree $T$ and a set of colors $C$, the linear relaxation $\mathcal{P}^*(T, C)$ of the standard model is an integer polytope.*

**Sketch of the proof.**

Given a fractional point $\hat{x} \in \mathcal{P}^*(T, C)$, we construct two points $\hat{x}^a, \hat{x}^b \in \mathcal{P}^*(T, C)$ in such a way that $\hat{x} = \frac{1}{2}(\hat{x}^a + \hat{x}^b)$. Then, $\hat{x}$ is not an extreme point of $\mathcal{P}^*(T, C)$ and hence every extreme point of $\mathcal{P}^*(T, C)$ is integer. □

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The following is a direct consequence of the previous results:

**Corollary**

Given a tree $T$ and a set of colors $C$, both the separation and the optimization problem over $\mathcal{PL}(T, C, L)$ can be solved in polynomial time for any list $L$.

# Standard polytope on block graphs

**Definition (Clique inequalities, Coll et al., 2002)**

Given a clique $K \subseteq V$ and color $c \in C$, the *clique inequality* associated to $K$ and $c$ is defined as

$$\sum_{i \in K} x_{ic} \leq 1. \quad (1)$$

**Theorem (Coll et al., 2002)**

*Clique inequalities (1) are valid for $\mathcal{P}(G, C)$ (and the inequalities obtained by using maximal cliques are facet-defining for $\mathcal{P}(G, C)$).*

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Some insights on block graphs:

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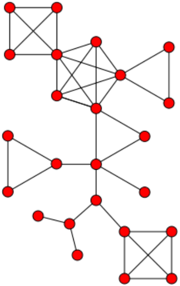

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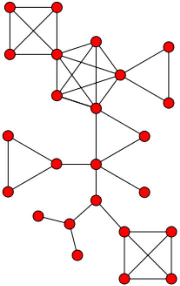

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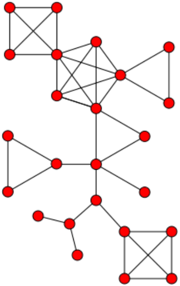

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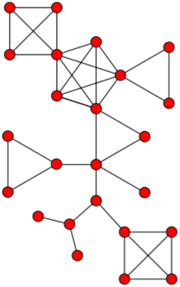

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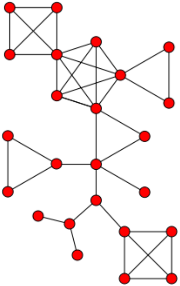

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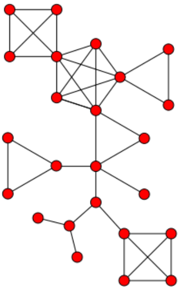

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The proof starts by proving that if *G* is just a clique, any fractional solution is a convex combination of other two solutions (as in the proof for trees but here the solutions need to fulfill some extra requirements). Then, given a fractional solution, an induction is made on the number of cliques of the graph in order to obtain some characteristic subsolutions. Finally, these subsolutions are convexely combined to obtain the original fractional solution. □

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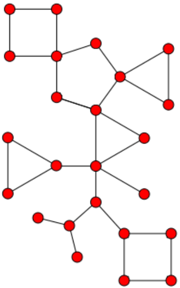

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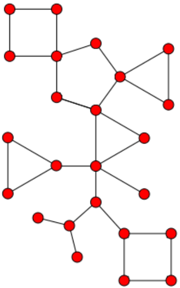

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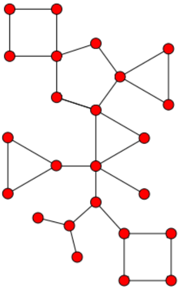

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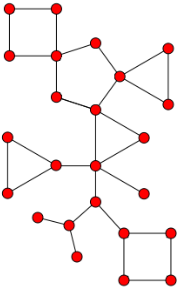

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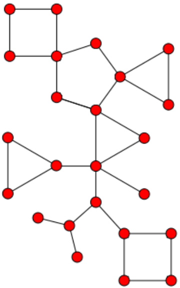

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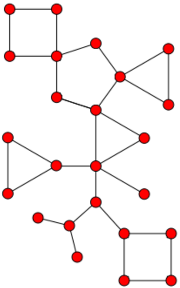

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# Outline

1. Introduction
   - Vertex coloring problems
   - Integer Programming models
   - Geometric algorithms (and implications...)
2. Standard IP formulation
   - The formulation and some general results
   - Trees, blocks and cacti graphs
3. Some final remarks and future work

# Some final remarks and future work

- The standard formulation is a very simple, easy to study, formulation with strong combinatorial properties.
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(Mehrotra & Trick, 1996)

Let's recall briefly the MIS model

$$\sum_{S: i \in S} x_S \geq 1 \qquad \forall i \in V$$
$$x_S \in \{0, 1\} \qquad \forall S \in \mathcal{S}(G)$$

We find some "interesting" polytopes for two families using the MIS formulation.

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Ok, now I've really finished the presentation...

Thank you again!