Group 7 - Maximum weighted independent set problem pondered by subgraphs

Given a graph $G = (V, E)$, consider the following independent set formulation for the Graph Coloring problem, where $\mathcal{S}$ is the set of all independent sets of G.

$$\min \quad \sum_{S \in \mathcal{S}} x_S \tag{1a}$$

$$\text{s.t.:} \quad \sum_{S : v \in S} x_S \geq 1, v \in V(G) \tag{1b}$$

$$x_S \in \{0, 1\}, \quad S \in \mathcal{S} \tag{1c}$$

This well-sutdied formulation is known for having solved some difficult graph coloring instances based on a branch-and-price algorithm. Its linear relaxation bound is know to be among the best of all formulations since it gives exactly the fractional chromatic number of the graph. The column generation for this problem is the problem of finding the maximum weighted independent set of G pondered by the dual costs of this formulation.

One way to strength even further the quality of the linear relaxation bound that seems natural (and has been used in other formulations) is too add constraints to this problem based on its subgraphs for which we know the exact chromatic number. Let us consider then a pool $\mathcal{H}$ of subgraphs. Then, we could add the following valid constraints:

$$\sum_{S : S \cap H \neq} x_S \geq \chi(H), \quad H \in \mathcal{H} \tag{2}$$

With such constraints we will have then another subproblem to generate the columns. Indeed this subproblem still is to find a maximum stable set, although in this case it is not pondered by the dual cost d_v associated with each of the node $v \in V$, but also by the dual costs d_H associated with each substructure $H \in \mathcal{H}$. Hence we have the following formulation, which comes directly from a classical adaptation of the MWIS formulation:

$$
\begin{aligned}
\max \quad & \sum_{v \in V(G)} d_v z_v + \sum_{v \in V(G)} d_H y_H && && \text{(3a)} \\
\text{s.t.:} \quad & x_u + x_v \leq 1, && (u,v) \in E(G) && \text{(3b)} \\
& y_H \leq \sum_{v \in H} z_v, && H \in \mathcal{H} && \text{(3c)} \\
& z_v \in \{0,1\}, && v \in V(G) && \text{(3d)} \\
& y_H \in \{0,1\}, && H \in \mathcal{H} && \text{(3e)}
\end{aligned}
$$

Having the problem well-defined it is important to note that we have not found until the present time any literature on this problem. So it is an wide open problem that would welcome any approach, at first. In our meeting we decided to enumerate some possibly viable research tracks that might interest the group.

- **Fast heuristic for column generation purposes.** Since the motivation of this study comes from the column generation algorithm for graph coloring, it is an interesting idea to develop a fast heuristic to find good solutions for the announced problem, in order to avoid solving the integer problem completely. It is known that for a minimization problem any column whoose cost is negative is sufficient for the continuation of the column generation. If a fast heuristic is used to generate a fast feasible solution of reasonable quality, it may be applied for the branch-and-cut-and-price method for graph coloring.

- **Polyhedral study of the problem.** Although this is a new problem, it may also be seen as an extension of the well-studied independent set problem. We could use results from IS and try to adaptat to our problem. One first approach is to lift known facets of the independent set polytope and a second possibility is to draw valid inequalities from it. We have worked on the maximal clique facets and shown that they could be lifted to facets of the new problem's polytope. A natural interest is to prove the same for any (or some) rank inequalities.

- **Efficient exact solution method for the problem.**

- **Limit the graph class or the subgraphs class.**