STIC AmSud Project
ParGO Team

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Here we are
Vertex coloring and correlated NP-Hard problems

**Graph Theory**
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- Cláudia L. Sales
- Rudini M. Sampaio
- Victor A. Campos

**Combinatorial Optimization**
- Carlos Diego Rodrigues
- Manoel B. Campêlo Neto
- Ricardo C. Corrêa
Graph Theoretic Approach

Stable Set Polytope of $G = (V, E)$: subset of vertices pairwise non-adjacent

vertex $u \in W$

vertex $v \notin W$
Graph Classes

NP-Hard variation of the vertex coloring problem + Graphs with special structural properties

↓

Polynomial time algorithms
A Typical Case: b-Coloring

Given a coloring \( c \), \( v \) is a b-vertex of color \( i \) if \( c(v) = i \) and \( v \) has at least one neighbor in every color class \( j, j \neq i \).

A b-coloring is a coloring such that each color class has a b-vertex.

Theorem

The Tight b-Chromatic Problem is \( NP \)-complete for connected bipartite graphs.

Theorem

The b-chromatic number of a split graph can be determined in polynomial time.
Graph Decomposition: a Powerful Tool

An example: Modular Decomposition (figure from Wikipedia)
Selected Publications and Problems

Combinatorial Optimization Approach

vertex $u \in W$
$x_W[u] = 1$

vertex $v \notin W$
$x_W[v] = 0$

Characteristic vector $x_W$ of $W \subseteq V$
Combinatorial Optimization Approach

\[ \text{vertex } u \in W \quad x_W[u] = 1 \]

\[ \text{vertex } v \notin W \quad x_W[v] = 0 \]

\textit{STAB}(G): Convex hull of characteristic vectors

For every \( x \in \text{STAB}(G) \),

\[ x[u] + x[v] \leq 1, \quad uv \in E \]
Maximum Stable Set (MSS) Problem

Linear program

$$\alpha(G) = \max \left\{ \sum_{u \in V} x[u] \mid x \in STAB(G) \right\}$$
Maximum Stable Set (MSS) Problem

Linear program

\[ \alpha(G) = \max \{ \sum_{u \in V} x[u] \mid x \in STAB(G) \} \]

Integer program

\[ \alpha(G) = \max \sum_{u \in V} x[u] \]

s.t. \[ x[u] + x[v] \leq 1, \quad uv \in E \]

\[ x[u] \in \{0, 1\}, \quad u \in V \]
Disjoint Stable Set Problems

Finding a family $\mathcal{W}$ of disjoint stable sets under certain constraints

Some examples

- MSS: maximize $|\bigcup \mathcal{W}|$ under $|\mathcal{W}| = 1$
- Maximum $k$-partite induced subgraph: maximize $|\bigcup \mathcal{W}|$ under $|\mathcal{W}| \leq k$
- Vertex coloring: minimize $|\mathcal{W}|$ when, for all $v \in V$, there exists $S \in \mathcal{W}$ such that $v \in S$
Covering Formulation for the Vertex Coloring Problem

\[ S_{max} - \text{family of maximal ss of } G \]

\[ \chi(G) = \min \sum_{S \in S_{max}} x[S] \]

s.t. \[ \sum_{S \in S_{max}: u \in S} x[S] \geq 1, \quad u \in V \]

\[ x[S] \in \{0, 1\}, \quad S \in S_{max} \]

Exponential number of variables

[Mehrotra, Trick 1996] - column generation implementation

[Hansen, Labbé, Schindl 2009] - polyhedral study;

\[ \sum_{S \in S_{max}} x[S] \geq \chi(G) \text{ if } G \text{ is critical and } \bar{G} \text{ is connected} \]
Representatives

Let us choose a representative for each nonempty stable set of $G$. One simple criterion is to choose the smallest vertex in the stable set according to a given order of the vertices [C, Campos, C 2008].
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Notation for the Representatives

Anti-neighborhood

$$\bar{N}^+(u) = \{v > u \mid uv \notin E\}$$

Subgraph induced by anti-neighbors

$$G^+(u) = G[\bar{N}^+[u] = \bar{N}^+(u) \cup \{u\}]$$

Stable sets represented by $u$

Stable sets of $G^+(u)$ containing $u$ itself
$x^u[u] = 1$ : $u$ is a representative

$x^u[v] = 1$ : $u$ represents $v \in \bar{N}^+[u]$
Formulations by Representatives

\[ \chi(G) = \min \left\{ \sum_{v \in V} x^v[v] \mid x \in \mathcal{X} \right\}, \text{ where } \mathcal{X} \text{ is given by} \]
Formulations by Representatives

$$\chi(G) = \min \left\{ \sum_{v \in V} x^v[v] \mid x \in \mathcal{X} \right\},$$ where $\mathcal{X}$ is given by

**Integer program**

$$x^u[v] + x^u[w] \leq x^u[u], u \in V, v, w \in \bar{N}^+(u), vw \in E$$

$$\sum_{u \in \bar{N}^-[v]} x^u[v] \geq 1, \quad v \in V$$

$$x^u[v] \in \{0, 1\}, \quad u \in V, v \in \bar{N}^+[u]$$
Formulations by Representatives

\[ \chi(G) = \min \left\{ \sum_{v \in V} x^v[v] \mid x \in \mathcal{X} \right\}, \text{ where } \mathcal{X} \text{ is given by} \]

**Integer program**

\[
\begin{align*}
x^u[v] + x^u[w] & \leq x^u[u], u \in V, v, w \in \bar{N}^+(u), vw \in E \\
\sum_{u \in \bar{N}^-[v]} x^u[v] & \geq 1, v \in V \\
x^u[v] & \in \{0, 1\}, \quad u \in V, v \in \bar{N}^+[u]
\end{align*}
\]

**“Linear” program**

\[
\begin{align*}
x^v & \in STAB_v(x^v[v](G^+(v)), \quad \sum_{u \in \bar{N}^-[v]} x^u[v] \geq 1, x^v[v] \in \{0, 1\}, v \in V
\end{align*}
\]
General Disjoint Stable Set Problem

\[
\text{max } f(V) = \sum_{v \in V} \sum_{w \in \bar{N}^+(v)} f^v[w] x^v[w] + e
\]

\[
a^v_1 x[v_1] \quad \ldots \quad a^v_n x[v_n]
\]

\[
x^V[V] \quad \ldots
\]

\[
-x^v_1 [v_1] \quad \ldots \quad -x^v_n [v_n]
\]

\[
c^v_1 x^v_1 [v_1] \quad \ldots \quad c^v_n x^v_n [v_n]
\]

\[
x^v_1 [\bar{N}^+(v_1)] \quad \ldots \quad x^v_n [\bar{N}^+(v_n)]
\]

\[
c^v_1 x^v_1 [\bar{N}^+(v_1)] \quad \ldots \quad c^v_n x^v_n [\bar{N}^+(v_n)]
\]

\[\text{subject to } \quad b x[V] \leq \ldots \leq 0 \quad \ldots \quad 0 \quad d\]
Selected Publications and Problems

Thank you