STIC AmSud Project
ParGO Team

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March 29, 2013

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Here we are

Vertex coloring and correlated NP-Hard problems

Graph Theory

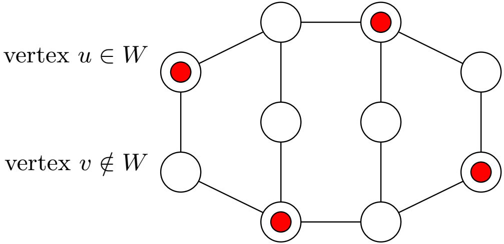
- Ana Shirley F. Silva
- Cláudia L. Sales
- Rudini M. Sampaio
- Victor A. Campos

Combinatorial Optimization

- Carlos Diego Rodrigues
- Manoel B. Campêlo Neto
- Ricardo C. Corrêa

Graph Theoretic Approach

Stable Set Polytope of $G = (V, E)$: subset of vertices pairwise non-adjacent



Graph Classes

NP-Hard variation of the vertex coloring problem

$+$

Graphs with special structural properties

$\Downarrow$

Polynomial time algorithms

A Typical Case: b-Coloring

- Given a coloring c, v is a b-vertex of color i if $c(v) = i$ and v has at least one neighbor in every color class j,$j \neq i$.
- A b-coloring is a coloring such that each color class has a b-vertex.

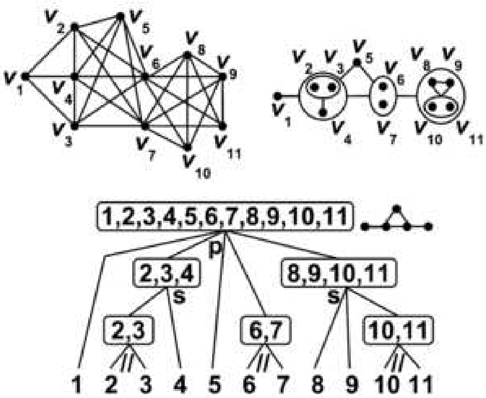
Theorem

The TIGHT B-CHROMATIC PROBLEM *is NP-complete for connected bipartite graphs.*

Theorem

The b-chromatic number of a split graph can be determined in polynomial time.

Graph Decomposition: a Powerful Tool

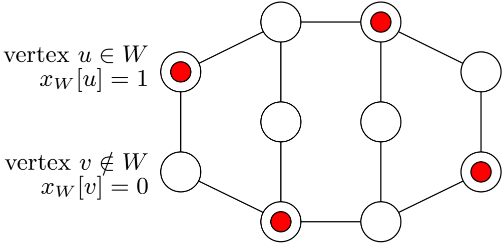


An example: Modular Decomposition (figure from Wikipedia)

Selected Publications and Problems

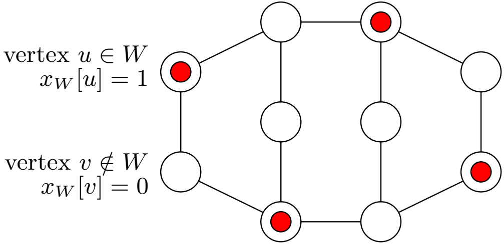
- J. Araújo, C. L. Sales, "On the Grundy number of graphs with few P_4's". *Disc. Appl. Math.*, 2012.
- M. Asté, F. Havet, C. L. Sales, "Grundy number and products of graphs". *Disc. Math.*, 2010.
- V. Campos, A. Gyárfás, F. Havet, C. L. Sales, F. Maffray, "New bounds on the Grundy number of products of graphs". *J. of Graph Theory*, 2012.
- V. Campos, S. Klein, R. Sampaio, A. Silva, "Two Fixed-Parameter Algorithms for the Cocoloring Problem", *ISAAC*, 2011.
- V. Campos, C. L. Sales, F. Maffray, A. Silva, "b-chromatic number of cacti". *LAGOS*, 2009.
- V. Campos, A. K. Maia, N. Martins, C. L. Sales, R. Sampaio, "Restricted Coloring Problems on graphs with few P_4's". *LAGOS*, 2011.
- F. Havet, C. L. Sales and L. Sampaio, "b-coloring of m-tight graphs". *Disc. Appl. Math.*, 2012.

Combinatorial Optimization Approach



Characteristic vector x_W of $W \subseteq V$

Combinatorial Optimization Approach



STAB(*G*): Convex hull of characteristic vectors

For every $x \in STAB(G)$,

$$x[u] + x[v] \leq 1, \quad uv \in E$$

Maximum Stable Set (MSS) Problem

Linear program

$$\alpha(G) = \max\{\sum_{u \in V} x[u] \mid x \in STAB(G)\}$$

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Integer program

$$\begin{aligned} \alpha(G) = \max \quad & \sum_{u \in V} x[u] & \\ \text{s.t.} \quad & x[u] + x[v] \leq 1, & uv \in E \\ & x[u] \in \{0,1\}, & u \in V \end{aligned}$$

Disjoint Stable Set Problems

Finding a family $\mathcal{W}$ of disjoint stable sets under certain constraints

Some examples

- MSS: maximize $|\cup \mathcal{W}|$ under $|\mathcal{W}| = 1$
- Maximum k-partite induced subgraph: maximize $|\cup \mathcal{W}|$ under $|\mathcal{W}| \leq k$
- Vertex coloring: minimize $|\mathcal{W}|$ when, for all $v \in V$, there exists $S \in \mathcal{W}$ such that $v \in S$

Covering Formulation for the Vertex Coloring Problem

$\mathcal{S}_{max}$ - family of maximal ss of G

$$\begin{aligned} \chi(G) = \min \quad & \sum_{S \in \mathcal{S}_{max}} x[S] \\ \text{s.t.} \quad & \sum_{S \in \mathcal{S}_{max} : u \in S} x[S] \geq 1, \quad u \in V \\ & x[S] \in \{0,1\}, \qquad S \in \mathcal{S}_{max} \end{aligned}$$

Exponential number of variables

[Mehrotra, Trick 1996] - column generation implementation

[Hansen, Labbé, Schindl 2009] - polyhedral study;

$\sum_{S \in \mathcal{S}_{max}} x[S] \geq \chi(G)$ if G is critical and $\bar{G}$ is connected

Representatives

Let us choose a representative for each nonempty stable set of G. One simple criterion is to choose the smallest vertex in the stable set according to a given order of the vertices [C, Campos, C 2008].

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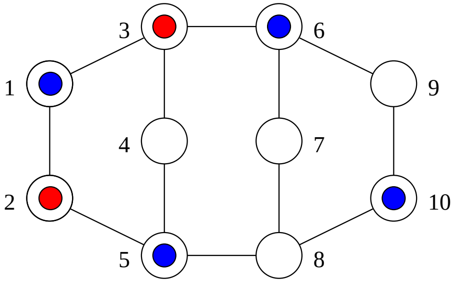

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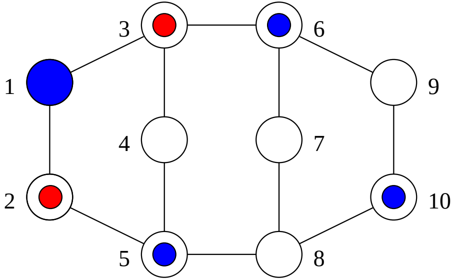

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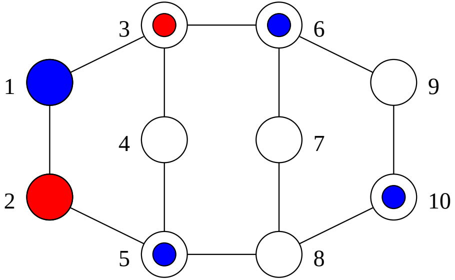
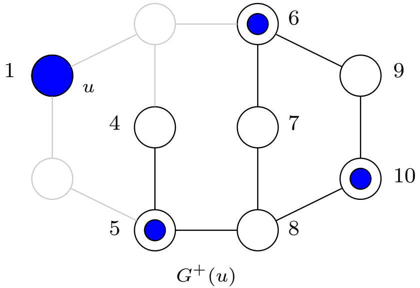


Notation for the Representatives



Anti-neighborhood

$$\bar{N}^+(u) = \{v > u \mid uv \notin E\}$$

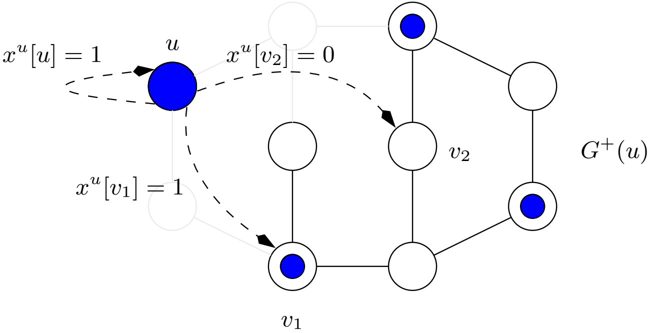
Subgraph induced by anti-neighbors

$$G^+(u) = G[\bar{N}^+[u] = \bar{N}^+(u) \cup \{u\}]$$

Stable sets represented by u

Stable sets of $G^+(u)$ containing u itself

Variables



$x^u[u] \in \{0, 1\}$: u is a representative

$x^u[v] \in \{0, 1\}$: u represents $v \in \bar{N}^+[u]$

Formulations by Representatives

$\chi(G) = \min\left\{\sum_{v \in V} x^v[v] \mid x \in \mathcal{X}\right\}$, where $\mathcal{X}$ is given by

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Integer program

$$
\begin{aligned}
& x^u[v] + x^u[w] \leq x^u[u], && u \in V, v, w \in \bar{N}^+(u), vw \in E \\
& \sum_{u \in \bar{N}^-[v]} x^u[v] \geq 1, && v \in V \\
& x^u[v] \in \{0,1\}, && u \in V, v \in \bar{N}^+[u]
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&x^u[v] \in \{0,1\}, && u \in V, v \in \bar{N}^+[u]
\end{aligned}
$$

"Linear" program

$$
x^v \in STAB_{x^v[v]}(G^+(v)), \sum_{u \in \bar{N}^-[v]} x^u[v] \geq 1, x^v[v] \in \{0,1\}, v \in V
$$

General Disjoint Stable Set Problem

$$\max f(V) = \sum_{v \in V} \sum_{w \in \bar{N}^+(v)} f^v[w] x^v[w] + e$$

$$\begin{array}{cccccc|c|c}
a^{v_1} x[v_1] & \cdots & a^{v_n} x[v_n] & & & & & b \\
x^V[V] & \cdots & & & & & & x[V] \\
-x^{v_1}[v_1] & & & x^{v_1}[\bar{N}^+(v_1)] & & & \leq & 0 \\
 & \ddots & & & \ddots & & & \ddots \\
 & & -x^{v_n}[v_n] & & & x^{v_n}[\bar{N}^+(v_n)] & & 0 \\
c^{v_1} x^{v_1}[v_1] & \cdots & c^{v_n} x^{v_n}[v_n] & c^{v_1} x^{v_1}[\bar{N}^+(v_1)] & \cdots & c^{v_n} x^{v_n}[\bar{N}^+(v_n)] & & d
\end{array}$$

Selected Publications and Problems

- M. C., V. Campos, R. C. C., "On the asymmetric representatives formulation for the vertex coloring problem," *DAM*, 2008.
- M. C., V. A. Campos, R. C. C., "Um Algoritmo de Planos-de-Corte para o Número Cromático Fracionário de um Grafo". *Pesquisa Operacional*, 2009.
- M. C., R. C. C., "A Combined Parallel Lagrangian Decomposition and Cutting-Plane Generation for Maximum Stable Set Problems". ISCO, 2010.
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- A. Basu, M. C., M. Conforti, G. Cornuéjols, G. Zambelli, "Unique Lifting of Integer Variables in Minimal Inequalities". *Math. Program.*, 2012.
- M. C., G. Cornuéjols, "Stable sets, corner polyhedra and the Chvátal closure". *Op. Res. Let.*, 2009.
- M. C., G. Cornuéjols, "The Chvátal closure of generalized stable sets in bidirected graphs". LAGOS, 2009.

Thank you